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Linear Measures, the Gini Index, and The Income–Equality Trade-off*

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This paper provides an axiomatization of linear inequality measures representing binary relations on the subspace of income profiles having identical total income. Interpreting the binary relation as a policymaker's preference, we extend the axioms to the whole space and find that they characterize linear social evaluation functions. The axiomatization seems to suggest that a policymaker who has a linear measure of inequality on a subspace should have a linear evaluation on the whole space. An extension of the preferences reflected in the Gini index to the whole space is represented by a linear combination of total income and the Gini index. *Journal of Economic Literature* Classification numbers: D30, D31, D60, D63, D81.

1. INTRODUCTION

This paper provides an axiomatization of linear evaluation functions[1] as a representation of a binary relation on profiles of incomes. In particular,

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[1] The term *linear* in the literature on income distribution means linear after arranging incomes in an increasing order. Formally, let $f \in R^n$, $f = (f_1, ..., f_n)$ be an income profile and let $\tilde{f}$ be the profile that is obtained from f by arranging the incomes in an increasing order. We say that a social evaluation function $J(f)$ is linear if there exists numbers $a_1, ..., a_n$ such that $J(f) = \sum_{i=1}^{n} a_i \tilde{f}_i$.

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the Gini index is obtained by a natural strengthening of the main axiom. The paper is closely related to the work on the measurement of inequality but the interpretation of the binary relation on the income profiles is different. Here the binary relation is interpreted as a preference order of society or a policymaker. Thus, if f and g are income profiles, $f \succ g$ means that if the policymaker had to choose between f and g she would choose f. While the literature on inequality measurement (notably Atkinson [2]) clearly recognizes the relationship between social welfare and inequality, its ultimate goal is to measure inequality per se. Thus, it attempts to find a mathematical representation for a binary relation $\succ^*$ with the following interpretation: $f \succ^* g$ means that f is more egalitarian than g. Clearly we would not expect that $\succ$ and $\succ^*$ coincide on the whole space of income profiles. For example, a typical assumption on $\succ^*$ is relative invariance which means that the level of inequality does not change when all incomes are multiplied by the same factor. When one considers the relation $\succ$, such a property is obviously unreasonable. However, if we restrict our attention to some subspace of income profiles where the total income is fixed, then it seems reasonable (or at least of interest) to assume that on such a subspace $\succ$ and $\succ^*$ do coincide. In other words, when the total income is fixed the policymaker determines her preferences according to her judgment about the level of inequality.[2] If this assumption is made our results can be interpreted as suggesting that if a policymaker has a linear measure of inequality on some subspace then she should have a linear evaluation on the whole space. In particular, the evaluation on the whole space can be represented as the sum of total income and an appropriate inequality index (i.e., a function that represents the preference on the subspace).

Specifically, we consider two domains:

1. the subspace of income distributions with fixed total income;
2. the whole space.

We first provide an axiomatization of a linear evaluation function on the subspace (the Gini index is obtained by a natural strengthening of the main axiom). Then we show that an "innocent" modification of one of the axioms implies a linear evaluation on the whole space. We do not see how this modification can be rejected while the original axiom is accepted. However, this, of course, is an intuitive claim and the reader will judge it for himself. The Gini index has acquired a special status and it is therefore of interest to derive the representation of a Gini index on the whole space. Specifically, let $\succcurlyeq$ be a preference order on the whole space with the

[2] For example, Ebert [6] (and in a different way Sheshinski [15]) studies preferences on the whole space that are derived from a preference order on pairs of total income and an inequality index, where the inequality index represents the preference on the subspace.

property that its restriction to some subspace is a Gini preference order (i.e., the Gini index represents the preference). Let $f = (f_1, ..., f_n)$ denote an income profile. If $\succcurlyeq$ still satisfies our axioms then it can be represented by a function $J(f)$ where $J(f) = \sum_{i=1}^{n} f_i - \delta \sum_{1 \leqslant i < j \leqslant n} |f_j - f_i|$ for some $0 < \delta < 1/(n-1)$. Thus, our result is that a Gini preference on the whole space can be represented by a linear combination of total income and the Gini index. Let us emphasize that, on the whole space, $J(f)$ is different from the Gini index, which is equivalent[3] to

$$\frac{1}{\sum_{i=1}^{n} f_i} \sum_{1 \leqslant i < j < n} |f_j - f_i|.$$

This of course is not surprising since the interpretations of $J(f)$ and the Gini index on the whole space are different. The Gini index is an inequality index; it can represent the preference of a decision maker only on the subspace where the total income is fixed. When two profiles that do not belong to the same subspace are compared, the total income should be taken into account as well.

Linear evaluation functions have been studied by Donaldson and Weymark [5], Meheran [9], Weymark [18], and Yaari [19], among others. (More general inequality measures were also discussed by Ebert [7].) Our axioms for the whole space are actually similar to those of Weymark and Yaari. In fact, our axiomatizations of a general linear functional on the whole space turn out to be a minor variation on a result by Weymark. (Yaari considers a model with a continuum of individuals; therefore, his proofs are different.)

However, all the previous work we are aware of deals with the whole space. By contrast, our focus is on a subspace (of fixed total income) and on the *relationship* between it and the whole space of income profiles. Thus, our results allow one to judge the reasonableness of linear functionals in general, and the Gini index in particular, as pure measures of inequality, neutralizing total-income effects. This, in turn, suggests some analogies between the measurement of inequality per se and the income–equality trade-off.

It should probably be made clear at the outset that we are not trying to defend linear evaluation functionals, nor to attack their appropriateness. The reader is likely to be a better judge of that. We simply note that the most salient drawback of linear measures is that the effect on the social welfare of a transfer of income from one individual to another depends only on the ranking of the incomes but not on their absolute levels. On the other hand, linear measures and in particular the Gini index are simple

[3] That is, when the size of the population is fixed.

functions, and as this paper and the work mentioned above show, they involve a simple set of assumptions. This simplicity is probably the main reason for the leading status of the Gini index in applications. In any case, our aim is to provide as-simple-as-possible axioms which are equivalent to the linear (or Gini) representation and to point out the conceptual linkage between the subspace and the whole space.

The paper is organized as follows. Section 2 contains a description of the axioms and the results. The main steps in the proofs are given in Section 3, while standard ones are relegated to an Appendix. Section 4 concludes.

2. Notation and Results

Let $N=\{1, ..., n)$ be the set of *individuals*. Let the *income profiles* (or simply "profiles") be

$$F=\{f\colon N\to \mathbf{R} \mid f(i)\geqslant 0\ \forall\ i\in N\},$$

to be identified with $\mathbf{R}^n_+$. In the sequel we will not distinguish between $f(i)$ and f_i.

Since we will be interested in subspaces across which total income is constant, it will prove useful to define, for $C\geqslant 0$,

$$F^C=\left\{f\in F \,\middle|\, \sum_{i\in N} f_i=C\right\}.$$

Some of our axioms will involve an assumption of order preservation, i.e., that two profiles do not change the income-ordering of individuals. In general, this condition (on pairs of profiles) is called comonotonicity (see Schmeidler [14]). However, since we will in anycase impose a symmetry axiom, it will facilitate notation to simply focus on monotone profiles. Define, then

$$F_M=\{f\in F \mid f_i\leqslant f_{i+1} \text{ for } 1\leqslant i\leqslant n-1\}$$

and

$$F^C_M=F^C\cap F_M.$$

For a permutation $\pi\colon N\to N$, and $f\in F$, define $\pi f\in F$ by $(\pi f)_i=f_{\pi(i)}$. Obviously, $\pi(F^C)=F^C$ for all $C\geqslant 0$. For every $f\in F$ define $f^{(\cdot)}=(f^{(1)}, f^{(2)}, ..., f^{(n)})$ to be the element of F_M for which there exists a permutation $\pi\colon N\to N$ satisfying $f^{(\cdot)}=\pi f$. Note that $f^{(\cdot)}$ is uniquely defined even if π is not.

For $f \in F$ and $i, j \in N$, i is said to *f-precede* j iff $f_i \leqslant f_j$ and there is no $k \in N$ for which $f_i < f_k < f_j$.

Let $\succcurlyeq \subseteq F \times F$ be a binary relation, to be interpreted as a *preference* relation. We will now formulate some axioms on $\succcurlyeq$. These axioms are parametrized by a set of profiles $H \subseteq F$; the theorems will be stated with the axioms required to hold for $H = F$, F^C, F_M or F_M^C.

A1 (H). *Weak Order.* $\succcurlyeq$ is complete, i.e., for every $f, g \in H$, $f \succcurlyeq g$ or $g \succcurlyeq f$, and transitive: for every $f, g, h \in H$, $f \succcurlyeq g$ and $g \succcurlyeq h$ imply that $f \succcurlyeq h$.

We use the regular notation of $\preccurlyeq = \succcurlyeq^{-1} = \{(f, g) \mid (g, f) \in \succcurlyeq\}$; $\sim = \succcurlyeq \cap \preccurlyeq$; $\succ = \succcurlyeq \backslash \sim$; $\prec = \preccurlyeq \backslash \sim$.

A2 (H). *Continuity.* For every $f \in H$ the sets $\{g \in H \mid g \succ f\}$, $\{g \in H \mid g \prec f\}$ are open in H. (That is, in the topology on H induced by the natural topology on R^n.)

A3 (H). *Symmetry.* For every $f, g \in H$, if there is a permutation $\pi: N \to N$ such that $g = \pi f$, then $f \sim g$.

A4 (H). *Monotonicity.* For every $f, g \in H$, if $f_i \geqslant g_i$ for all $i \in N$ and $f_j > g_j$ for some $j \in N$, then $f \succ g$.

The next two axioms all require some consistency between choices. They have a flavor of Savage's sure-thing principle (Savage [11]), but they also implicitly presuppose that utility is linear in income.

A5 (H). *Order-Preserving Gift.* For every $f, g, f', g' \in H \cap F_M$ and $i \in N$, if $f_j = f_j'$ and $g_j = g_j'$ for all $j \neq i$, and $f_i' = f_i + t$, $g_i' = g_i + t$ for some $t \in \mathbf{R}$, then $f \succcurlyeq g$ iff $f' \succcurlyeq g'$.

A5 says that the preference between two profiles of f and g, which agree on the social income-ordering, should not change if the same individual i receives a "gift" t in both f and g, provided that the resulting profiles f' and g' respect the same ordering. The logic behind it, which one may accept or reject, can best be seen if one first considers the cases it excludes: if, for instance, f and g do not agree on the social income-ordering, individual i may be the poorest in f and the richest in g. Increasing his income would therefore have a different effect on the inequality in f and in g, and the preference between them may well change. Similarly, even if both f and g are monotone, a gift of t to individual i may make him richer than $(i+1)$ in f, but leave him poorer than $(i+1)$ in g. Again, this asymmetric impact on inequality may give rise to preference reversal.

It is only in the cases where the above do not happen that A5 can be invoked to deduce that preference reversal should not occur.

Note that for $H \subseteq F^C$ A5 is vacuously satisfied, since for $t \neq 0$, f' and g' do not belong to F^C if f and g do. Hence, we will need a total-income preserving version of it, which will deal with transfers (from one individual to another) rather than gifts. However, we formulate it in a potentially stronger form, as explained below.

A6 (H). *Order-Preserving-Transfer.* For all $f, g, f', g' \in H$ and $i, j \in N$, if the following hold,

(i) i f-, g-, f'-, and g'-precedes j;

(ii) $f_i' = f_i + t \qquad g_i' = g_i + t$
$f_j' = f_j - t \qquad g_j' = g_j - t$ for some $t > 0$; and

(iii) $f_k' = f_k \qquad g_k' = g_k$ for $k \notin \{i, j\}$,

then $f \succcurlyeq g$ iff $f' \succcurlyeq g'$.

To better understand A6, let us first consider the case of $H = F_M^C$ (as will be done in Theorem A below). For this H, A6 is the "natural" reformulation of A5 when one is restricted to a constant-total-income hyperplane. Indeed, A5 (F_M) implies A6 (F_M^C), as is easily verified.

However, we will also use A6 for $H = F^C$ (in Theorems B and D), in which case it makes a stronger claim: starting out with some $f, g \in F^C$, f' preseves the order of f, as does g' with respect to g. But A6 requires that there be no preference reversal even if f and g do *not* agree on the social ordering. In particular, the pair (i, j) may be the poorest in f and the richest in g, yet a transfer from j to i should not, according to A6 (F^C), change the preference between f and g. The distinction between general linear welfare functions on F^C and the more specific Gini index will be whether A6 is required to hold on F_M^C or on all of F^C.

A7 (H). *Inequality Aversion.* For all $f, f' \in F_M \cap H$ and $1 \leqslant i < n$, if

$$f_j' = f_j, \qquad \text{for all} \quad j \notin (i, i+1)$$

$$f_i' = f_i + t, f_{i+1}' = f_{i+1} - t, \qquad \text{for some} \quad t > 0,$$

then $f' \succ f$.

A7 simply states that a transfer of money from an individual to the next richest one, in such a way that the social income-ordering is preserved, will result in a strictly preferred social profile. Thus, A7 is a weak version of the famous Dalton–Pigou principle which states that a transfer of money from a rich person to a poor person, which leaves the rich person richer, will reduce inequality.

We will say that "$\succcurlyeq$ satisfies An on H" if $\succcurlyeq$ satisfies $An(H)$.

We can finally formulate our main results:

THEOREM A. *For every $C>0$ and every $\succcurlyeq \subseteq F^C \times F^C$ the following are equivalent*:

(i) $\succcurlyeq$ *satisfies* A1, A2, A3, *and* A7 *on* F^C, *and* A6 *on* F^C_M.

(ii) *there is a vector* $p=(p_1, ..., p_n)$ *with* $p_1>p_2> \cdots >p_n$ *such that for all* $f, g \in F^C$,

$$f \succcurlyeq g \Leftrightarrow \sum_{i \in N} p_i f^{(i)} \geqslant \sum_{i \in N} p_i g^{(i)}.$$

Furthermore, in this case the vector p (*in* (ii)) *is unique up to a positive affine transformation.*

THEOREM B. *For every $C>0$ and every $\succcurlyeq \subseteq F^C \times F^C$ the following are equivalent*:

(i) $\succcurlyeq$ *satisfies* A1, A2, A3, A6, *and* A7 *on* F^C;

(ii) *for all* $f, g \in F^C$,

$$f \succcurlyeq g \Leftrightarrow \sum_{1 \leqslant i<j \leqslant n} |f_i-f_j| \leqslant \sum_{1 \leqslant i<j \leqslant n} |g_i-g_j|.$$

THEOREM C. *For every $\succcurlyeq \subseteq F \times F$ the following are equivalent*:

(i) $\succcurlyeq$ *satisfies* A1, A2, A3, A4, A5, *and* A7 *on* F;

(ii) *there is a vector* $p=(p_1, ..., p_n)$ *with* $p_1>p_2> \cdots >p_n>0$ *such that for all* $f, g \in F$,

$$f \succcurlyeq g \Leftrightarrow \sum_{i \in n} p_i f^{(i)} \geqslant \sum_{i \in n} p_i g^{(i)}.$$

Furthermore, in this case the vector p (*in* (ii)) *is unique up to multiplication by a positive scalar.*

THEOREM D. *For every $\succcurlyeq \subseteq F \times F$, the following are equivalent*:

(i) $\succcurlyeq$ satisfies A1, A2, A3, A4, A5, A6, *and* A7 *on* F;

(ii) *There is a number* δ, $0<\delta<1/(n-1)$, *such that for all* $f, g \in F$,

$$f \succcurlyeq g \Leftrightarrow \sum_{i=1}^{n} f_i - \delta \sum_{1 \leqslant i<j \leqslant n} |f_i-f_j| \geqslant \sum_{i=1}^{n} g_i - \delta \sum_{1 \leqslant i<j \leqslant n} |g_i-g_j|.$$

Furthermore, in this case the coefficient δ (*in* (ii)) *is unique.*

Theorem A provides a characterization of linear evaluation functions on the subspace. Theorem B states that the Gini index is obtained if, in addition to the assumptions in Theorem A, we require that the order-preserving transfer axiom apply to every pair of profiles f and g (and not only to pairs of profiles that agree on the social ordering). Theorems C and D are the counterparts of Theorems A and B, respectively, when the whole space of income profiles is considered and when we add A5 (F). Thus, Theorem C characterizes a linear evaluation function on the whole space,[4] while Theorem D provides the representation of an extension of the preferences, reflected in the Gini index, to the whole space. Note that the preference on the whole space is represented by a linear combination of total income and the Gini index.

We now want to suggest that a decision maker that satisfies A6 (F_M^C) on the subspace should satisfy A5 (F) on the whole space. As we noted, A5 (F) implies A6 (F_M^C) because a transfer from i to j can be obtained by a gift to j and a "negative" gift to i. Thus, A6 (F_M^C) is obtained by putting a certain restriction on the application of A5 (F), namely, that a gift to one person should be offset by a negative gift to another person. We do not see why a decision maker would accept A5 (F) with the restriction but not without it. One could, for example, object to A5(F) on the grounds that the effect of a given gift to individual i on social welfare should depend on total income (and not only on the rank of the individual involved). However, if the decision maker satisfies A6 (F_M^C) on the subspace, it means that the effect of a given transfer between two individuals on the evaluation function depends only on the rank of the individuals involved but not on their absolute level of income. Hence, our reponse to the above objection is that it seems inconsistent for a decision maker to evaluate a change in individual i's income according to total income, but not to take into account i's own income. We hope that our results are of interest whether the above view is accepted or not. However, if this view is accepted then the implication is that a decision maker who has a linear evaluation function on the subspace should have a linear evaluation on the whole space as well. In particular, if the preference of the decision maker on the subspace corresponds to her evaluation of inequality then the results can be interpreted as follows: if a decision maker has a linear inequality measure on some subspace then she should have a linear evaluation function on the whole space. In particular, if the decision maker evaluates inequality according to the Gini index, then her evaluation on the whole space is a linear combination of total income and the Gini index.

[4] Theorem C is a simple corollary of Theorem 3 in Weymark [18]. We are grateful to an anonymous referee for pointing this out.

3. Proofs and Related Analysis

3.1. *Proof of Theorem* A

A few words on the strategy of the proof may be in order. Our main tool is axiom A6, which guarantees that an order-preserving transfer of income t from j to i, where i (immediately) precedes j, does not alter the preference between two profiles. We will first extend this condition and show that it applies even if i and j are not consecutive in the income ordering (Lemma 3.1.1), and that any transfer vector whose addition is order-preserving may be added to two profiles without reversing the preference between them (Lemma 3.1.4).

As explained in more detail in the sequel (following the proof of Lemma 3.1.4), this result will allow us to define the "substitution rate" between transfers from j to i and transfers from l to k for some $i, j, k, l \in N$. This rate, denoted by σ_{ijk_l}, will measure how many dollars should be transferred from j to i to have the same equality impact as a single-dollar transfer from l to k.

These substitution rates, which will be shown to be well behaved, will give rise to the linear representation. Many of the proofs which follow are standard, if not straightforward. We therefore relegate some of the more tedious ones to the appendix. In these cases we provide here the "Idea of Proof."

We now turn to the proof. First, note that, in view of A3, it suffices to provide a vector $p = (p_1, ..., p_n)$ such that for all $f, g \in F_M^C$,

$$f \succcurlyeq g \Leftrightarrow \sum_{i \in N} p_i f_i \geqslant \sum_{i \in N} p_i g_i.$$

We first need some auxiliary results, which will strengthen the main axiom, i.e., A6 (F_M^C). It will prove useful to focus on the interior of F_M^C,

$$(F_M^C)^0 = \{f \in F^C \mid 0 < f_1 < f_2 < \cdots < f_n\}.$$

All the following lemmata and claims in this subsection are steps in the proof of (i) $\Rightarrow$ (ii) and presuppose (i).

We will now show that a transfer from individual j to i, which respects comonotonicity, does not induce preference reversal.

Lemma 3.1.1. *For all $f, g, f', g' \in (F_M^C)^0$, and all $i, j \in N$, $i \neq j$ if*

$$f'_k = f_k, \qquad g'_k = g_k \qquad \text{for all} \quad k \notin \{i, j\}$$

and for some $t \in R$

$$f'_i = f_i + t \qquad g'_i = g_i + t$$
$$f'_j = f_j - t \qquad g'_j = g_j - t,$$

then

$$f \succcurlyeq g \quad \text{iff} \quad f' \succcurlyeq g'.$$

Idea of Proof. To transfer t from j to i, say, when $j > i$, one may transfer t from j to $(j-1)$, from $(j-1)$ to $(j-2)$ and so forth. However, such transfers may lead us out of F_M^C. Thus, instead of transferring all of t at once, one may make repeated transfers of small enough ε. ∎

Remark 3.1.2. Note that for F_M^C the same result cannot be similarly proven. Consider, for instance, $f = (0, 1, 2)$ and $f' = (1, 1, 1)$. Although one can obtain f' from f by a single order-preserving transfer from 3 to 1, no (finite) sequence of order-preserving transfers between "adjacent" individuals would yield f' from f.

Obviously, an "infinite" sequence will do the trick, i.e., f' can be obtained as the limit of f_n, where each f_n can be obtained from f by a finite sequence of adjacent transfers. However, starting from $f \succ g$, continuity of $\succcurlyeq$ only guarantees $f' \succcurlyeq g'$, which is not sufficient for our purposes.

This is the main reason to focus on $(F_M^C)^0$ (rather than F_M^C) first. Only when enough structure is proven to exist in the preference over $(F_M^C)^0$ will we use continuity to derive the representation on its boundary as well.

We will also need the corresponding extension of A7 :

LEMMA 3.1.3. *Let* $f, f' \in (F_M^C)^0$, *where for some* $1 \leqslant i < j \leqslant n$ *and some* $t > 0$,

$$f_i' = f_i + t \qquad f_j' = f_j - t$$

and

$$f_k' = f_k \qquad \textit{for all} \quad k \notin \{i, j\},$$

then $f' \succ f$.

Proof. As in Lemma 3.1.1, by successive applications of A7 and transitivity. ∎

A further extension of A6 is the following:

LEMMA 3.1.4. *Let* $f, g, f', g' \in (F_M^C)^0$, *where* $f' = f + t$, $g' = g + t$ *for some* $t \in \mathbf{R}^n$. *Then* $f \succcurlyeq g$ *iff* $f' \succcurlyeq g'$.

Idea of Proof. By induction on the maximal index i such that i is involved in the transfers. ∎

Equipped with these tools, we now turn to the main step of the proof. The general strategy is as follows: for every triple (i, j, t), where $1 \leqslant i < j \leqslant n$ and $t \in \mathbf{R}$, consider the "improvement" obtained if, in a given profile, a transfer of t from j to i takes place. We will show that these triples can be ranked (in terms of the "size" of improvement) regardless of the base profile. We will further show that this binary relation is homogeneous, i.e., that the improvement in (i, j, t) is "greater" than that implied by the transfer (k, l, s) iff the same holds for $(i, j, \alpha t)$ and $(k, l, \alpha s)$ for $\alpha > 0$.

This homogeneity will give rise to coefficients σ_{ijkl}, which will provide the substitution rate between transfers from j to i and transfers from l to k. We will show that a given profile is equivalent to a profile generated from it by offsetting transfers (according to these substitution rates).

Next we will use these coefficients σ_{ijkl} to define the "weights" p_i and will show that the weighted avarage $J(F) = \sum_i p_i f_i$ is also unaltered when offsetting transfers are made. Finally, each profile f will be "normalized" (in some appropriate sense) by a sequence of offsetting transfers and it will only remain to show that $J(\bullet)$ represents $\succcurlyeq$ on the normalized profiles.

Let us begin by using the following notation: for $f \in F^C$, $i, j \in N$, and $t \in \mathbf{R}$, let $f_{(i,j,t)} \in \mathbf{R}^n$ be given by $f_{(i,j,t)} = f - te^j + te^i$, where e^i is the i-th unit vector.

Denote $T = \{(i, j, t) \mid 1 \leqslant i < j \leqslant n,\ t \in \mathbf{R}\}$.

LEMMA 3.1.5. *Let $f, g \in F^C$, and (i, j, t), $(k, l, s) \in T$ be such that*

$$f_{(i,j,t)},\ f_{(k,l,s)},\ g_{(i,j,t)},\ g_{(k,l,s)} \in (F_M^C)^0.$$

Then, $f_{(i,j,t)} \succcurlyeq f_{(k,l,s)}$ iff $g_{(i,j,t)} \succcurlyeq g_{(k,l,s)}$.

Proof. Let $\tilde{t} = g - f$, and use Lemma 3.1.4 (with $\tilde{f} = f_{(i,j,t)}$, $\tilde{g} = f_{(k,l,s)}$, $\tilde{f}' = g_{(i,j,t)}$, $\tilde{g}' = g_{k,l,s)}$). ∎

In view of this lemma , we will write $(i, j, t) \succcurlyeq (\succ, \sim)\, (k, l, s)$ iff there exists $f \in F^C$ such that $f_{(i,j,t)} \succcurlyeq (\succ, \sim)\, f_{(k,l,s)}$ and both are members of $(F_M^C)^0$.

LEMMA 3.1.6. *Let (i, j, t), $(k, l, s) \in T$ with $t, s > 0$, and assume that for some $f, g \in F^C$, $f_{(i,j,t)}, g_{(i,j,-t)}, f_{(k,l,s)}, g_{(k,l,-s)} \in (F_M^C)^0$. Then $(i, j, t) \succcurlyeq (k, l, s)$ iff $(i, j, -t) \preccurlyeq (k, l, -s)$.*

Proof. Define $h = (g_{(i,j,-t)})_{(k,l,-s)} \in F^C$ and note that

$$g_{(i,j,-t)} = h_{(k,l,s)} \qquad \text{and} \qquad g_{(k,l,-s)} = h_{(i,j,t)}.$$

Considering h and f and applying Lemma 3.1.5 one obtains the desired conclusion. ∎

LEMMA 3.1.7. *Suppose that for* (i, j, t), $(k, l, s) \in T$ *and* $\alpha > 0$ *there exist* $f, g \in (F_M^C)^0$ *such that* $f_{(i,j,\alpha t)}$, $f_{(k,l,\alpha s)}$, $g_{(i,j,t)}$, $g_{(k,l,s)} \in (F_M^C)^0$. *Then* $(i, j, t) \succcurlyeq (k, l, s)$ iff $(i, j, \alpha t) \succcurlyeq (k, l, \alpha s)$.

(*Note that in the statement above the existence of f and g is only required to guarantee that the triples involved are comparable.*)

Idea of Proof. The main step is to prove the claim for natural α by induction, using 3.1.4. The extension to rational and then irrational α is standard. ∎

As in the above proof we conclude that for every $i < j$, $k < l$ and for every small enough $t > 0$ there is an $\bar{s} > 0$ such that $(i, j, t) \sim (k, l, \bar{s})$. Furthermore, for $s < \bar{s}$ we have $(i, j, t) \succ (k, l, s)$ and for $s > \bar{s}$ $(k, l, s) \succ (i, j, t)$. Since, moreover, for $\alpha > 0$, $\alpha\bar{s}$ would correspond to αt, we define

$$\sigma_{ijkl} = t/\bar{s} > 0.$$

σ_{ijkl} is a substitution rate of sorts; it measures how much "money" should be transferred from j to i to have the same (equality) impact as a single-unit transfer from l to k.

Conclusion 3.1.8. For all (i, j, t), $(k, l, s) \in T$, with $t, s > 0$, if there is $f \in (F_M^C)^0$ such that $f_{(i,j,t)}$, $f_{(k,l,s)} \in (F_M^C)^0$, then $(i, j, t) \succcurlyeq (\succ, \sim)(k, l, s)$ iff $t \geqslant (>, =)\, \sigma_{ijkl} s$.

To simplify notation, we extend $\succcurlyeq$ to all of T, using 3.1.8 (together with 3.1.3 and 3.1.6) as the definition of $\succcurlyeq$ if such an f does not exist. (For instance, if $t > C$.)

We will need two properties of the substitution rates σ_{ijkl}:

LEMMA 3.1.9. *For all* $i, j, k, l, r, q \in N$ *with* $i < j$; $k < l$; $r < q$,

$$\sigma_{ijkl}\sigma_{klrq} = \sigma_{ijrq}.$$

Proof. Take $t, s, u > 0$ such that $(i, j, t) \sim (k, l, s) \sim (r, q, u)$. Then

$$\sigma_{ijkl} = t/s$$

$$\sigma_{klrq} = s/u$$

and since $(i, j, t) \sim (r, q, u)$

$$\sigma_{ijrq} = t/u = (t/s)(s/u) = \sigma_{ijkl}\sigma_{klrq}. \quad ∎$$

LEMMA 3.1.10. *For all* i, j, k, r, l *with* $i < j$, $r < k < l$,

$$\sigma_{ijrk} + \sigma_{ijkl} = \sigma_{ijrl}.$$

Proof. Let $t, s, u, v > 0$ satisfy

$$(i, j, t) \sim (r, k, s)$$
$$(i, j, u) \sim (k, l, s)$$
$$(i, j, v) \sim (r, l, s)$$

whence

$$\sigma_{ijrk} = t/s$$
$$\sigma_{ijkl} = u/s$$
$$\sigma_{ijrl} = v/s$$

Further assume w.l.o.g. (without loss of generality) that all of t, u, s, v are small enough so that there is an $f \in F^C$ for which $f_{(i,j,t)}$, $f_{(i,j,u)}$, $f_{(i,j,v)}$, $f_{(i,j,t+u)}$, $f_{(r,k,s)}$, $f_{(k,l,s)}$, $f_{(r,l,s)} \in (F_M^C)^0$.

Then we have

$$f_{(i,j,t)} \sim f_{(r,k,s)}$$

and, by Lemma 3.1.1,

$$(f_{(i,j,t)})_{(i,j,u)} \sim (f_{(r,k,s)})_{(i,j,u)}.$$

However, since $(i, j, u) \sim (k, l, s)$,

$$(f_{(r,k,s)})_{(i,j,u)} \sim (f_{(r,k,s)})_{(k,l,s)},$$

which implies that

$$(f_{(i,j,t)})_{(i,j,u)} \sim (f_{(r,k,s)})_{(k,l,s)}$$

or

$$f_{(i,j,t+u)} \sim f_{(r,l,s)},$$

whence $t + u = v$.

Finally, $t/s + u/s = v/s$ and

$$\sigma_{ijrk} + \sigma_{ijkl} = \sigma_{ijrl}. \quad \blacksquare$$

We can finally define the weights p_i for $i \in N$: let $p_1 = 0$, $p_2 = -1$ and for $2 < k \leqslant n$, $p_k = -\sigma_{121k}$. (So that

$$\sigma_{121k} = (p_1 - p_k)/(p_1 - p_2).)$$

Next define, for all $f \in F^C$,

$$J(f) = \sum_{k=1}^{n} p_k f_k.$$

We now wish to show that equivalent transfers have identical effect on J:

LEMMA 3.1.11. *Let (i, j, t), $(k, l, s) \in T$ satisfy $(i, j, t) \sim (k, l, s)$. Then for all $f \in F^C$ such that $f_{(i,j,t)}, f_{(k,l,s)} \in (F^C_M)^0$,*

$$J(f_{(i,j,t)}) = J(f_{(k,l,s)}).$$

Idea of Proof. Calculation using 3.1.9 and 3.1.10. ∎

As for the converse:

LEMMA 3.1.12. *Suppose that for some $f \in (F^C_M)^0$ and (i, j, t), $(k, l, s) \in T$ such that $f_{(i,j,t)}, f_{(k,l,s)} \in (F^C_M)^0$, $J(f_{(i,j,t)}) = J(f_{(k,l,s)})$. Then $(i, j, t) \sim (k, l, s)$.*

Proof. By the computations of the previous proof one obtains

$$t\sigma_{12ij} = s\sigma_{12kl},$$

i.e.,

$$t/s = \sigma_{12kl}/\sigma_{12ij} = \sigma_{ijkl},$$

which suffices by 3.1.8. ∎

We are approaching the final steps of the proof. It will be useful, however, to have explicit mention of the following:

LEMMA 3.1.13. *Suppose that (i, j, t), (k, l, s) satisfy $(i, j, t) \sim (k, l, s)$, and assume that $f \in F$ satisfies*

$$f, f_{(i,j,t)}, f_{(k,l,s)}, (f_{(i,j,t)})_{(k,l,-s)} \in (F^C_M)^0.$$

Then $f \sim (f_{(i,j,t)})_{(k,l,-s)}$.

Proof. Since $f_{(i,j,t)} \sim f_{(k,l,s)}$, we may use Lemma 3.1.1 to obtain

$$(f_{(i,j,t)})_{(k,l,-s)} \sim (f_{(k,l,s)})_{(k,l,-s)} = f. \quad ∎$$

The next step is to show that for every $f \in (F^C_M)^0$ there is an $\hat{f} \in (F^C_M)^0$ such that $f \sim \hat{f}$ and $J(f) = J(\hat{f})$, where $\hat{f}$ is normalized in some sense. If we could choose $\hat{f}$ from F^C_M, we would like it to be of the form $(\alpha, \alpha, ..., \alpha, \beta)$, where $\beta > \alpha$, and then to show, for the unique α and β determined by $\hat{f} \in F^C$ and $J(\hat{f}) = J(f)$, that $\hat{f}$ can be obtained from f by a sequence of pairs of offsetting transfers (i, j, t), $(k, l, -s)$, where $t = \sigma_{ijkl} s$.

However, given that our results are guaranteed to hold for $(F_M^C)^0$, we have to choose $\hat{f}$ which is strictly monotone. This does not make a fundamental difference, though it complicates both the statement and the proof.

LEMMA 3.1.14. *Given $f \in (F_M^C)^0$, there is an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ there are $\alpha = \alpha(\varepsilon)$ and $\beta = \beta(\varepsilon)$ such that $\hat{f}_\varepsilon \sim f$ and $J(\hat{f}_\varepsilon) = J(f)$, where $\hat{f}_\varepsilon \in (F_M^C)^0$ is defined by*

$$(\hat{f}_\varepsilon)_i = \alpha + (i-1)\varepsilon \qquad \text{for} \quad i < n$$

$$(\hat{f}_\varepsilon)_n = \beta.$$

Furthermore, $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are given by

$$\alpha = \left[\sum_{i=1}^{n-1} p_i - (n-1) p_n \right]^{-1} \times \left[J(f) - p_n C - \varepsilon \sum_{i=2}^{n-1} (i-1) p_i + \varepsilon p_n \frac{(n-1)(n-2)}{2} \right]$$

$$\beta = \left[\sum_{i=1}^{n-1} p_i - (n-1) p_n \right]^{-1} \times \left[C \sum_{i=1}^{n-1} p_i - (n-1) J(f) - \varepsilon \frac{(n-1)(n-2)}{2} \sum_{i=1}^{n-1} p_i + \varepsilon(n-1) \sum_{i=2}^{n-1} (i-1) p_i \right].$$

Idea of Proof. First, one shows that for small enough ε, $\hat{f}_\varepsilon$ is indeed in $(F_M^C)^0$. Next, one uses induction to construct, for $k \leqslant n-1$, a vector f_ε^k, such that $f_\varepsilon^k \sim f$, $J(f_\varepsilon^k) = J(f)$, and the first k individuals are richer than their preceding ones by ε exactly. The derivation of f_ε^{k+1} from f_ε^k is done by transferring some money from $k+1$ to each of $1, \ldots, k$ and transferring the rest to the richest individual n, in such a way that preference (and J-value) is preserved. ∎

We can finally return to F_M^C, including its boundary:

LEMMA 3.1.15. *For every $f \in F_M^C$ there is $\hat{f} \in F_M^C$ with the following properties:*

(i) $\hat{f}_i = \alpha$ *for* $i < n$, $\hat{f}_n = \beta$ *for some* $\beta \geqslant \alpha$;

(ii) $\hat{f} \sim f$; *and*

(iii) $J(\hat{f}) = J(f)$.

Proof. Starting with $f \in (F_M^C)^0$, we obtain $\hat{f}_\varepsilon$, for every $\varepsilon \in (0, \bar{\varepsilon})$ by 3.1.14. Letting ε tend to zero, the explicit formulae of 3.1.14 show that $\hat{f}_\varepsilon$ converge to some $\hat{f} \in F_M^C$, satisfying (i). Condition (ii) would follow from the continuity of $\succcurlyeq$, while (iii) follows from the continuity of the (linear) functional J.

As for $f \in F_M^C \backslash (F_M^C)^0$, let $f_n \in (F_M^C)^0$ satisfy $f_n \to f$, and let $\hat{f}_n$ be the corresponding profile for f_n. Given the explicit formulae of 3.1.14 we know that $\hat{f}_n \to \hat{f}$ where $\hat{f}$ satisfies (i). By continuity of J, (iii) also holds. Finally, since $\succcurlyeq$ is continuous, it is also closed (as a subset of $\mathbf{R}^{2n}$), and since $f_n \sim \hat{f}_n$, $f \sim \hat{f}$. ∎

It therefore suffices to show that J represents $\succcurlyeq$ on the one-dimensional half-space $\{(\alpha, \alpha, ..., \alpha, \beta) \mid (n-1)\alpha + \beta = C,\ \beta \geqslant \alpha \geqslant 0\}$. Indeed, we have

LEMMA 3.1.16. *Let $\alpha, \beta, \gamma, \delta$ satisfy $\beta \geqslant \alpha \geqslant 0$, $\delta \geqslant \gamma \geqslant 0$,*

$$(n-1)\alpha + \beta = (n-1)\gamma + \delta = C.$$

Then the following are equivalent:

(i) $\beta - \alpha < \delta - \gamma$;

(ii) $J((\alpha, \alpha, ..., \alpha, \beta)) > J((\gamma, \gamma, ..., \gamma, \delta))$;

(iii) $g \equiv (\alpha, \alpha, ..., \alpha, \beta) \succ (\gamma, \gamma, ..., \gamma, \delta) \equiv f$.

Idea of Proof. Standard continuity arguments. ∎

Employing symmetry of $\succcurlyeq$, this concludes the proof of Theorem A. ∎

3.2. *Proof of Theorem* B

Given Theorem A, we know that for every $f, g \in F^C$

$$f \succcurlyeq g \text{ iff } \sum_{i \in N} p_i f^{(i)} \geqslant \sum_{i \in N} p_i g^{(i)}$$

for some $p_1 > p_2 > \cdots > p_n$. We now further assume that A6 holds on all of F^C.

LEMMA 3.2.1. *For every* $1 \leqslant i \leqslant n-1$,

$$p_i - p_{i+1} = p_1 - p_2.$$

Proof. Assume w.l.o.g. $i > 1$. Choose $f \in (F_M^C)^0$. Let $\varepsilon > 0$ satisfy $\varepsilon < (1/2)(f_{i+1} - f_i)$ for all $1 \leqslant i \leqslant n-1$, and define $f' = f + \varepsilon e^1 - \varepsilon e^2 \in (F_M^C)^0$. Define a permutation $\pi\colon N \to N$ as follows:

(i) if $i>2$, $\pi(1)=i$ $\pi(2)=i+1$
$\pi(i)=1$ $\pi(i+1)=2$ and
$\pi(k)=k$ for $k\notin\{1,2,i,i+1\}$.

(ii) if $i=2$, $\pi(1)=2$ $\pi(2)=3$ $\pi(3)=1$ and
$\pi(k)=k$ for $k\notin\{1,2,3\}$.

Define $g=\pi f$, and note that in g, individuals 1 and 2 are ranked as the i- and $(i+1)$-poorest, respectively. Finally, define $g'=g+\varepsilon e^1-\varepsilon e^2$.

By symmetry, $f\sim g$. However, individual 1 f-, f'-, g- and g'-precedes 2, and A6(F^C) implies that $f'\sim g'$. Hence

$$J(f')-J(f)=J(g')-J(g),$$

where J is defined as in Section 3.1. This implies

$$\varepsilon(p_1-p_2)=\varepsilon(p_i-p_{i+1})$$

which completes the proof of the lemma. ∎

To complete the proof of Theorem B, recall that for every $a>0$ and $b\in\mathbf{R}$ the vector $q=(q_1,\dots,q_n)$ defined by $q_i=ap_i+b$ also satisfies

$$f\succcurlyeq g \text{ iff } \sum_{i\in N} q_i f^{(i)}\geqslant\sum_{i\in N} q_i g^{(i)}$$

for all $f,g\in F^C$. Setting

$$a=2(n-1)/(p_1-p_n)$$

and

$$b=(n-1)[1-2p_1/(p_1-p_n)]$$

yields

$$q_i=n+1-2i$$

for $1\leqslant i\leqslant n$. (Note that this makes use of the fact that $p_i-p_{i+1}=p_1-p_2$ for all i.)

On the other hand,

$$-\sum_{1\leqslant i<j\leqslant n}|f_i-f_j|=\sum_{i\in N}(n+1-2i)f^{(i)},$$

and the theorem is proven. ∎

3.3. *Proof of Theorem C.*

This theorem is a variant of Theorem 3 of Weymark [18], and so is its proof. Weymark's axioms are (1) symmetry—our A3; (2) weak order—A2; (3) continuity—equivalent to A2; and (4) "weak independence of income source," which follows directly from our A5. Thus, his Theorem 3 (p. 419) implies that there exists a vector $p=(p_1, ..., p_n)$ such that

$$f \succcurlyeq g \Leftrightarrow \sum_{i\in N} p_i f^{(i)} \geqslant \sum_{i\in N} p_i g^{(i)}.$$

Imposing A4 implies that $p_i > 0$ for all $i \in N$, while A7 guarantees that $p_i > p_{i+1}$ for all $i < n$. ∎

3.4. *Proof of Theorem* D

Given Theorem C, we know that for all $f, g \in F$

$$f \succcurlyeq g \text{ iff } \sum_{i\in N} p_i f^{(i)} \geqslant \sum_{i\in N} p_i g^{(i)}$$

with $p_1 > p_2 > \cdots > p_n > 0$. Furthermore, A6 is known to hold on F, and in particular also on F^C for all $C > 0$. Thus, by Lemma 3.3.1, $p_i - p_{i+1} = p_1 - p_2$ for all $1 \leqslant i \leqslant n-1$.

Setting

$$\alpha = 2/(p_1 + p_n) > 0$$

and

$$\delta = \frac{1}{(n-1)} \frac{p_1 - p_n}{p_1 + p_n},$$

it is straightforward to verify that for all $1 \leqslant i \leqslant n$

$$\alpha p_i = 1 + (n - 2i + 1)\delta.$$

It only remains to note that, for all $f \in F$,

$$\sum_{i\in N} [1 + (n-2i+1)\delta] f^{(i)} = \sum_{i\in N} f_i - \delta \sum_{1\leqslant i<j\leqslant n} |f_i - f_j|$$

and that $\delta < 1/(n-1)$ follows from $p_n > 0$. ∎

4. Concluding Remarks

1. The functionals discussed in this paper are linear on cones of vectors which agree on the income ordering of individuals, i.e., cones of "comonotonic" vectors in the sense of Schmeidler [14]. Indeed, these functionals turn out to be Choquet integrals with respect to some non-additive measures. (See Choquet [3].)

Thus, the measurement of inequality literature is closely related to the recent developments in the theory of choice under uncertainty (as opposed to risk). In particular, the axiomatizations of Choquet-integral representations in the context of uncertainty may be reinterpreted to derive income-inequality measures with non(-necessarily-)linear utility functions. The reader is referred to the seminal papers of Schmeidler [12–14], as well as Gilboa [8], Wakker [17], and Sarin and Wakker [10].

While these axiomatizations are formulated for spaces which correspond to our space F, some of them may be adapted to a subspace F^C. For instance, Schmeidler's axiomatization, which uses a mixture space (as in Anscombe and Aumann [1] and as opposed to Savage [11]), may be used to derive a representation on F^C, which is closed under convex combinations.

While our Theorem A required a lengthy proof, we find it is using much more intuitive axioms than Schmeidler's "comonotonic independence." However, our main point could also be conveyed using his result: the "same" axioms which are used to derive a linear functional on a subspace F^C will also yield a similar representation on all of F. It seems hard to defend the axioms on the subspace while rejecting them on the whole space.

2. It is sometimes convenient to model a population as a continuum of agents endowed with a σ-algebra, say, $[0, 1]$ with the Borel sets. All our axioms will have natural counterparts if one assumes a nonatomic σ-additive measure on these Borel sets to be given in the model. Symmetry is then required to hold with respect to the group of measurable and measure-preserving permutations, and the continuity of $\succcurlyeq$ should be stipulated with respect to convergence in the measure. In this topology one may approximate every profile by a simple profile, which is constant on every element of an equi-measure finite partition. With the representation theorems obtained above, the derivation of similar theorems for this setup is then straightforward.

3. It is easy to verify that, in each of the theorems, the axioms are independent. We omit the simple examples.

4. John Weymark suggested an alternative proof of Theorem A which will first use a classical result (such as Debreu [4]) for a numerical representation and then go on to show that the representing functional is linear. We find our constructive proof more intuitive, or at least more elementary. Admittedly, this is a matter of taste.

5. Finally, note that Theorem A provides a linear-representation on F^C only for inequality-averse preferences. This condition, however, is not crucial. We used it since it eliminates some non-insightful complications from the proof, while being natural in our context.

5. Appendix: Proofs

PROOF OF LEMMA 3.1.1. W.l.o.g. assume that $i<j$. Furthermore, w.l.o.g. we can also assume that $t>0$, for if $t<0$ one can switch the roles of f and f', g and g'.

For $h \in (F_M^C)^0$, let

$$d(h) = \min_{1 \leqslant i < n} (h_{i+1} - h_i).$$

Choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2} \min\{d(f), d(f'), d(g), d(g')\}$. For integers $i \leqslant k \leqslant j$ and $0 \leqslant r \leqslant \lfloor t/\varepsilon \rfloor \equiv M$ (where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x), define

$$f_{r,k} = f - r\varepsilon e^j + (r-1)\, \varepsilon e^i + \varepsilon e^k$$
$$g_{r,k} = g - r\varepsilon e^j + (r-1)\, \varepsilon e^i + \varepsilon e^k,$$

where

$$e^l \in \mathbf{R}^n \qquad \text{satisfies} \qquad (e^l)_l = 1,\ (e^l)_s = 0 \quad \text{for } s \neq l.$$

Further, let $\delta = t - M\varepsilon \geqslant 0$ and define

$$f_{M+1,k} = f - (M\varepsilon + \delta) e^j + M\varepsilon e^i + \delta e^k$$
$$g_{M+1,k} = g - (M\varepsilon + \delta) e^j + M\varepsilon e^i + \delta e^k.$$

Note that for all integers $i \leqslant k \leqslant j$ and $0 \leqslant r \leqslant M+1$, $f_{r,k}, g_{r,k} \in (F_M^C)^0$, and that

$$f_{1,j} = f; \qquad g_{1,j} = g;$$
$$f_{M+1,i} = f'; \qquad g_{M+1,i} = g';$$

and

$$f_{r,i} = f_{r+1,j}; \qquad g_{r,i} = g_{r+1,j} \qquad \text{for} \quad r \leqslant M.$$

Finally, for $k > i$, A6(F_M^C) implies that

$$f_{r,k} \succcurlyeq g_{r,k} \Leftrightarrow f_{r,k-1} \succcurlyeq g_{r,k-1}$$

for all $0 \leqslant r \leqslant M+1$, whence the result follows. ∎

Proof of Lemma 3.1.4. We will prove the following claim for all $0 \leqslant k \leqslant n$. For all $f, g, f', g' \in (F_M^C)^0$ such that $f' = f + t$, $g' = g + t$ for some $t \in \mathbf{R}^n$ and $t_i = 0$ for all $1 \leqslant i \leqslant k$, $f \succcurlyeq g$ iff $f' \succcurlyeq g'$. Note that for $k = 0$ this is the desired result.

The proof is by induction on $(n-k)$. For $k=n-1$ or $k=n$ the claim is trivial since $f'=f$ and $g'=g$. Assume, then, that the claim was proven for $k=r+1$ and consider the case $k=r$.

Let there be given, then, f, f', g, g', and t as in the claim. Note that $f_i'=f_i$ and $g_i'=g_i$ for $1\leqslant i\leqslant r$. Assume w.l.o.g. that $t_{r+1}<0$. (If $t_{r+1}>0$, reverse the roles of f and f', g and g'.)

Define $\hat{f}$, $\hat{g}$ as follows:

$$\begin{array}{lll} \hat{f}_i=f_i=f_i' & \hat{g}_i=g_i=g_i' & \text{for} \quad i\leqslant r \\ \hat{f}_{r+1}=f'_{r+1} & \hat{g}_{r+1}=g'_{r+1} & \\ \hat{f}_i=f_i & \hat{g}_i=g_i & \text{for} \quad r+1<i<n \\ \hat{f}_n=f_n-t_{r+1} & \hat{g}_n=g_n-t_{r+1}. & \end{array}$$

It is easily verifiable that $\hat{f}, \hat{g}\in(F_M^C)^0$. Furthermore, $f\succcurlyeq g$ iff $\hat{f}\succcurlyeq\hat{g}$ by Lemma 3.1.1. However, by the claim for $k=r+1$, $\hat{f}\succcurlyeq\hat{g}$ iff $f'\succcurlyeq g'$. ∎

Proof of Lemma 3.1.7. First consider the case $t, s>0$. Let us begin with $\alpha=r\in\mathbb{N}$, and prove by induction on r.

Assume that $f\in(F_M^C)^0$ and (i, j, t), $(k, l, s)\in T$ are such that

$$f_{(i,j,t)}, f_{(k,l,s)}, f_{(i,j,rt)}, f_{(k,l,rs)}\in(F_M^C)^0.$$

(Note that if f and g are given as in the lemma, for $\alpha>1$, $t, s>0$ we also have $f_{(i,j,t)}, f_{(k,l,s)}\in(F_M^C)^0$.)

Define, for $0\leqslant v, \mu\leqslant r$ with $v+\mu\leqslant r$,

$$h_{v,\mu}=(f_{(i,j,vt)})_{(k,l,\mu s)}.$$

Note that

$$\begin{array}{ll} h_{v,\mu}\in(F_M^C)^0 & \text{for all} \quad v,\mu\geqslant 0 \quad \text{with} \quad v+\mu\leqslant r \\ h_{0,0}=f & \\ h_{r,0}=f_{(i,j,rt)}, & h_{0,r}=f_{(k,l,rs)}. \end{array}$$

Further observe that, for $0\leqslant v\leqslant r-1$ and $\mu=r-v\geqslant 1$,

$$h_{v,\mu}=(h_{v,\mu-1})_{(k,l,s)}; \qquad h_{v+1,\mu-1}=(h_{v,\mu-1})_{(i,j,t)},$$

whence,

$$h_{v+1,\mu-1}\succcurlyeq h_{v,\mu} \text{ iff } (i, j, t)\succcurlyeq(k, l, s),$$

by definition of the latter relation, setting $f=h_{v,\mu-1}$. By transitivity we also obtain

$$h_{r,0}\succcurlyeq h_{0,r} \text{ iff } (i, j, t)\succcurlyeq(k, l, s).$$

We therefore conclude that for any positive integer α—hence for any rational α—$(i, j, \alpha t) \succcurlyeq (k, l, \alpha s)$ iff $(i, j, t) \succcurlyeq (k, l, s)$, whenever there are f and g as in the provisions of the lemma.

Next consider irrational $\alpha > 0$. It will here be useful to distinguish indifference from strict preference. If $(i, j, t) \sim (k, l, s)$, then for every rational α (for which the involved triples are comparble) $(i, j, \alpha t) \sim (k, l, \alpha s)$ and the conclusion follows by continuity of $\succcurlyeq$. If, however, $(i, j, t) \succ (k, l, s)$, we invoke Lemma 3.1.3 to deduce that $(k, l, s) \succ (i, j, 0)$ whence, again by continuity, there is $\hat{t} \in (0, t)$ such that $(i, j, \hat{t}) \sim (k, l, s)$. Therefore, $(i, j, \alpha\hat{t}) \sim (k, l, \alpha s)$. However, Lemma 3.1.3 also implies that $(i, j, \alpha t) \succ (i, j, \alpha\hat{t})$. Hence, $(i, j, t) \succcurlyeq (k, l, s)$ iff $(i, j, \alpha t) \succcurlyeq (k, l, \alpha s)$ for all positive α.

We can now turn to the cases in which s, t, or both are not strictly positive. Obviously, by 3.1.3 again, if $t > 0 \geqslant s$ or $t \geqslant 0 > s$, for all $i < j$ and $k < l$ $(i, j, t) \succ (k, l, s)$ and the same holds for αt, αs where $\alpha > 0$. Finally, the case $s, t < 0$ follows from $s, t > 0$ in view of Lemma 3.1.6. ∎

Proof of Lemma 3.1.11. Note that

$$J(f_{(i,j,t)}) = J(f) + t(p_i - p_j) = J(f) + t(\sigma_{121j} - \sigma_{121i})$$

(with $\sigma_{1211} = 0$).

Similarly,

$$J(f_{(k,l,s)}) = J(f) + s(\sigma_{121l} - \sigma_{121k}).$$

However, by Lemma 3.1.10,

$$\sigma_{121j} - \sigma_{121i} = \sigma_{12ij}$$

$$\sigma_{121l} - \sigma_{121k} = \sigma_{12kl}.$$

Hence,

$$J(f_{(i,j,t)}) = J(f) + t\sigma_{12ij}$$

$$J(f_{(k,l,s)}) = J(f) + s\sigma_{12kl}.$$

However, $t = s\sigma_{ijkl}$ and, by Lemma 3.1.9,

$$t\sigma_{12ij} = s\sigma_{ijkl}\sigma_{12ij} = s\sigma_{12kl}. \quad ∎$$

Proof of Lemma 3.1.14. It is straightforward to check that should $\hat{f}_\varepsilon$ belong to F^C and satisfy $J(\hat{f}_\varepsilon) = J(f)$, α and β can be computed from these two equations and should equal the expressions above.

These horrendous expressions are given here explicitly for two reasons. First, we must convince the reader that this system has a unique solution (which is obvious since $p_i > p_j$ for $i < j$ in view of A7), and that for small enough ε it will be in $(F_M^C)^0$. To this end, note that—again, since p_i are monotonically decreasing—$J(f)=\sum_{i=1}^n f_i p_i > p_n c$. Hence, $\alpha > 0$ for small ε. Further,

$$\beta-\alpha=\left[\sum_{i=1}^{n-1} p_i-(n-1)p_n\right]^{-1}\left[C\sum_{i=1}^{n} p_i-nJ(f)\right]+O(\varepsilon),$$

and—once more by monotonicity of p_i—the first two terms are positive.

The second reason will become clear later on, when we shrink ε to zero and claim convergence of $\hat{f}_\varepsilon$.

Let $\bar{\varepsilon}$ be small enough, then, to guarantee that $\hat{f}_\varepsilon \in (F_M^C)^0$ and to satisfy $\bar{\varepsilon}<(f_i-f_{i-1})$ for all $1<i\leqslant n$.

Considering $\varepsilon \in (0,\bar{\varepsilon})$, we will show that for every $1\leqslant k\leqslant n-1$ there exists f_ε^k such that (i) $(f_\varepsilon^k)_i-(f_\varepsilon^k)_{i-1}=\varepsilon$ for $2\leqslant i\leqslant k$; (ii) $(f_\varepsilon^k)_i-(f_\varepsilon^k)_{i-1}\geqslant\varepsilon$ for $k<i\leqslant n$; (iii) $f_\varepsilon^k \sim f$; and (iv) $J(f_\varepsilon^k)=J(f)$. The proof is by induction on k, and the existence of $f_\varepsilon^{n-1}=\hat{f}_\varepsilon$ will complete the proof of the lemma.

For $k=1$, condition (i) is vacuous and we may take $f_\varepsilon^1=f$. Assume, then, that f_ε^k was found and consider f_ε^{k+1} (for $k\leqslant n-2$).

If $(f_\varepsilon^k)_{k+1}-(f_\varepsilon^k)_k=\varepsilon$, we may choose $f_\varepsilon^{k+1}=f_\varepsilon^k$. Assume then that

$$(f_\varepsilon^k)_{k+1}-(f_\varepsilon^k)_k=\varepsilon+t, \qquad \text{for} \quad t>0.$$

One may obtain f_ε^{k+1} from f_ε^k by the following process: we will make an identical transfer $s>0$ from $k+1$ to each of $1,\ldots,k$, and a transfer of $r>0$ from $k+1$ to n, such that these transfers will offset each other. However, in order to use Lemmata 3.1.5 and 3.1.11 (which will guarantee the preservation of properties (iii) and (iv)) we need to split this transfer into a sequence of pairs of offsetting transfers.

First, we set

$$r=t\left[\sum_{i=1}^{k}(p_i-p_{k+1})+(k+1)(p_{k+1}-p_n)\right]^{-1}\sum_{i=1}^{k}(p_i-p_{k+1})$$

$$s=t\left[\sum_{i=1}^{k}(p_i-p_{k+1})+(k+1)(p_{k+1}-p_n)\right]^{-1}(p_{k+1}-p_n).$$

Note that $r, s>0$, and that

$$r+(k+1)s=t$$

$$r(p_{k+1}-p_n)=s\sum_{i=1}^{k}(p_i-p_{k+1}).$$

The first equation guarantees that the difference $(f_\varepsilon^k)_{k+1}-(f_\varepsilon^k)_k$ will be decreased by t exactly. The second one—that the overall transfers will preserve the value $J(f_\varepsilon^k)$.

We now split these transfers as follows: for $1 \leqslant i \leqslant k$, define

$$r_i = (p_i - p_{k+1})/(p_{k+1} - p_n)s.$$

Note that $r_i > 0$ and $\sum_{i=1}^{k} r_i = r$. Next define $f_\varepsilon^{k,0} = f_\varepsilon^k$ and for $1 \leqslant i \leqslant k$,

$$(f_\varepsilon^{k,i}) = ((f_\varepsilon^{k,i-1})_{(k+1-i, k+1, s)})_{(k+1, n, -r_{k+1-i})}.$$

That is, we first transfer r_k from $k+1$ to n and s from $k+1$ to k. Only then do we transfer r_{k-1} from $k+1$ to n, and s from $k+1$ to $k-1$, and so forth.

Thus, $f_\varepsilon^{k,i} \in (F_M^C)^0$ for $0 \leqslant i \leqslant k$. Furthermore,

$$(k+1-i, k+1, s) \sim (k+1, n, r_{k+1-i})$$

by 3.1.12. Hence, by 3.1.14, $f_\varepsilon^{k,i} \sim f_\varepsilon^{k,i-1}$ and $J(f_\varepsilon^{k,i}) = J(f_\varepsilon^{k,i-1})$ for all $i \leqslant k$. In particular, for $i = k$ we obtain f_ε^{k+1} satisfying (i)–(iv), and this completes the proof of the lemma. ∎

Proof of Lemma 3.1.16. The equivalence of (i) and (ii) follows from the definition of J, combined with the observation that $p_i > p_j$ for $i < j$. To see that (i) implies (iii), consider $f_\varepsilon \in F^C$ defined by

$$(f_\varepsilon)_i = \gamma + (i-1)\varepsilon \qquad 1 \leqslant i < n$$
$$(f_\varepsilon)_n = \delta - \tfrac{1}{2}(n-1)(n-2)\varepsilon.$$

If ε is small enough for $f_\varepsilon \in F_M^C$, f_ε can be obtained from f by successive adjacent transfers of size ε. Thus, by A7, $f_\varepsilon \succ f$. Choose $\varepsilon > 0$ such that $(f_\varepsilon)_n > \beta$ and $(f_\varepsilon)_{n-1} < \alpha$.

Similarly, for $\rho > 0$ consider g_ρ defined by

$$(g_\rho)_i = \alpha + (i-1)\rho \qquad 1 \leqslant i < n$$
$$(g_\rho)_n = \beta - \tfrac{1}{2}(n-1)(n-2)\rho.$$

Consider ρ small enough so that $g_\rho \in F_M^C$. Since $f_\varepsilon, g_\rho \in (F_M^C)^0$, g_ρ may be obtained from f_ε by successive adjacent transfers. Hence, $g_\rho \succ f_\varepsilon$. Letting ρ tend to zero, one concludes that $g \succcurlyeq f_\varepsilon \succ f$, and (iii) is proven. Finally, since $\beta - \alpha = \delta - \gamma$ immediately implies that $f \sim g$, (iii) $\Rightarrow$ (i) follows. ∎

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