Aggregation of Multiple Prior Opinions

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May 27, 2010

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- Different utilities (Impossibility by Mongin, 1995)
- Manipulability (Impossibility by Gibbard-Satterthwaite, 1973)

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- Can be derived from a unanimity condition a la Harsanyi (1955)
- Can be interpreted as a Bayesian belief if one expert knows the true probability, and λ is the DM's belief about which expert it is.

Knightian Uncertainty/Ambiguity

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- A rich history: Shafer (1986) argues that Bernoulli (1713) had non-additive probabilities in mind for court cases
- Main point: it is not about bounded rationality. It is not necessarily more rational to pretend that one has probabilities than to admit that one doesn't.

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- Maxmin EU: there exists a set of probabilities C such that

$$V(f) = \min_{P \in C} \int_S u(f(s))\, dP(s)$$

We do not discuss here...

- Nau (2006), Klibanoff-Marinacci-Mukerji (2005): "smooth preferences"

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- We assume the simple maxmin model.

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- If $\Lambda = \Delta(\{1, ..., n\})$ the DM is maxmin over the experts

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- Implied by a unanimity condition of sorts

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- Doesn't suffice for our purposes

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- The axiom requires this to hold for several f_i's and any convex weight vector

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- ... not if she is uncertainty averse: f offers state-by-state insurance, which is valuable when the DM doesn't know which expert is right.

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- Mixture operation: for every $P, Q \in L$ and every $\alpha \in [0,1]$, $\alpha P + (1-\alpha)Q \in L$ is given by

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- $f_R \in F$ is the constant act given by $f_R(s) = R$ for all $s \in S$
- F_c – all constant acts.
- Each of $\succsim_i$ is maxmin EU: there exist a non-constant utility $u_i : X \to \mathbb{R}$ and a convex and closed set of measures on S, C_i, such that

  $$f \succsim_i g \quad \text{iff} \quad J_i(f) \geq J_i(g)$$

  where

  $$J_i(f) = J_{u_i, C_i}(f) \equiv \min_{p \in C_i} \int u(f) dp. \tag{3}$$

  Moreover, (u_i, C_i) are the unique pair that represents $\succsim_i$ as in (3) (up to an increasing affine transformation of u_i).

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- Meaning: a bit of richness. Note that full dimensionality is not required.

The Main Axiom: Expert Uncertainty Aversion (EUA)

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- Stated in terms of the valuations J_i – can be translated to certainty equivalents

Main Result

Theorem

- *The following three conditions are equivalent:*

  (i) $(\succsim_i)_{i=0}^n$ *satisfy* **EUA**

  (ii) *There is a convex and closed set* $\Lambda \subseteq \Delta(\{1,\ldots,n\})$ *such that*

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- However, any Λ that satisfies (ii) satisfies (iii) and vice versa