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A Combination of Expected Utility and Maxmin Decision Criteria

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A new explanation of the Allais paradox is suggested. A decision rule representable by an increasing function of the expected utility and the minimum utility is axiomatized. The lexicographic separability of this function is also discussed and axiomatized.

1. INTRODUCTION

1.1. *Motivation*

Since the publication of Allais (1953), presenting the famous "Allais paradox," the Expected Utility (EU) theory of von-Neumann–Morgenstern (vNM) (1947) has been the subject of numerous generalizations designed to cope with the difficulties it raised (and with other difficulties as well).

It is beyond the scope of this paper to provide even the briefest survey of these generalizations, some of the most recent of which are Kahneman and Tversky (1979), Machina (1982), Quiggin (1982), Yaari (1987), Segal (1984), Chew (1984), Fishburn (1985, and many others), and Dekel (1986) (see Machina (1987) for a survey of the literature).

This paper suggests yet another generalization of EU theory which may explain the Allais paradox. Our purpose is to provide an axiomatically based model which satisfies the following requirements:

(a) It is not too general, i.e., not more than necessary in order to explain the Allais paradox and its variations. The model should not allow violations of EU theory which are not supported by empirical results (nor by intuitive reasoning) as those supporting the Allais paradox.

(b) It is general enough to explain minor variations of the paradox. More specifically, it will be contended that the Allais paradox challenges the continuity

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axiom (as well as the independence axiom) and that the model should explain violations of this axiom as well.

(c) It is simple with regard to the axioms as well as the decision rule. The axioms should be natural and appealing; the decision rule should be easy to understand and compute.

Of course, the author does not purport to have fully achieved these goals. But it seems that the model presented here satisfies the above requirement to a reasonable extent.

Remark. This model is very close to the one presented in Jaffray (1986). The models have been developed independently.

1.2. *The Main Argument*

Let us consider a simple variation of the Allais example, which is to be found in Kahneman and Tversky (1979), where it is referred to as "the certainty effect":

Consider the four lotteries, A, B, C, D defined by

$$\begin{array}{ll} A(4000)=0.8, & A(0)=0.2 \\ B(3000)=1, & \\ C(4000)=0.2, & C(0)=0.8 \\ D(3000)=0.25, & D(0)=0.75. \end{array}$$

("$P(x)=\alpha$" means that the lottery P assigns probability α to the consequence x. Here the consequences are the gain of x dollars.)

The empirical results reported in Kahneman and Tverskey (1979) show that a prevalent preference relation is $A<B$, $C>D$. However, if we introduce another lottery E such that $E(0)=1$, we may observe that

$$0.25A+0.75E=C$$

$$0.25B+0.75E=D$$

so that the preferences listed above biolate the independence axiom of vNM.

Let us now multiply all prizes by 1000. It seems reasonable, although this conjecture has not been tested empirically, that a similar example may be found whenever A assigns any positive probability to the \$0 prize. In particular, if we set

$$A_\alpha(4{,}000{,}000)=1-\alpha, \qquad A_\alpha(0)=\alpha$$

$$B(3{,}000{,}000)=1$$

it should not surprise us to observe a preference relation satisfying $A_\alpha<B$ for all $\alpha>0$ and $A_0>B$. This is a contradiction to the continuity axiom.

(Those readers who refuse to consider such preferences as "rational" should replace the \$0 prize by the consequence of death. It seems that at least in this case the preferences shown above cannot be dismissed as "irrational.")

We now wish to consider a simple decision rule which is consistent with these preferences. Let $u(x) = x$ be a utility function. For each lottery P, let $w(P)$ represent the worst consequence to which P assigns positive probability, and define a functional

$$F(P) = E(u(P)) + u(w(P)).$$

It is easily verifiable that the preference relation induced by F satisfies the preferences discussed above.

The intuitive justification for such a decision rule is "the certainty effect," that is, the special role certainty plays in decision making under uncertainty. A "pessimistic" individual using this rule evaluates lotteries both by their expected utility and by their minimum utility. (The roots of the notion of a "pessimistic" individual who wishes to maximize the minimal possible payoff trace back to Wald (1950).) The decision criterion formulated above is an example of a combination of the EU and Maxmin criteria.

1.3. *The Axioms*

Considering the vNM axioms again, we would like to have a simple modification of them which will be consistent with the decision-making patterns discussed above.

Since we restrict our attention to the Allais paradox (and we do not try to cope with other anomalies such as intransitivity of preferences), we retain the assumption that the preference relation is a weak order.

The continuity axiom will, of course, have to be weakened. It states that whenever $P < Q < R$ there are $\alpha, \beta \in (0, 1)$ such that

$$\alpha P + (1-\alpha) R < Q < \beta P + (1-\beta) R.$$

We will accept this axiom only when the worst consequences of P and Q are equivalent. If they fail to meet this requirement, the sheer mixture of them may involve discontinuities of preferences, and there is no reason to believe the axiom should hold. However, preferences are allowed to be discontinuous in the sense of violating the vNM axiom only when the mixture of lotteries makes the certain uncertain.

The independence axiom states that $P \geqslant R$ if and only if

$$\alpha P + (1-\alpha) Q \geqslant \alpha R + (1-\alpha) Q$$

for any $\alpha \in (0, 1)$ and any lottery Q. We will adopt this axiom only when it is guarenteed that the mixture of lotteries does not result in a biased change of the worst consequences of the lotteries under comparison. More specifically, we will assume that the axiom will hold if $w(P)$ and $w(R)$ are equivalent (but not necessarily otherwise). In this case, the worst consequences of the mixed lotteries $[\alpha P + (1-\alpha) Q]$ and $[\alpha R + (1-\alpha) Q]$ are also equivalent (for all $w(Q)$).

These axioms are sufficient for partial results, but they are also consistent with the preferences defined by the functional

$$F(P) = E(u(P)) - 2u(w(P)).$$

This functional implies that $x > y$ for any two consequences such that $u(x) < u(y)$. This is obviously an overgeneralization of the theory, and we will use another axiom, the meaning of which is monotonicity (or "stochastic dominance"): whenever a consequence in a certain lottery is replaced by a preferred one, the modified lottery is preferred to the original one. A simple formulation of this axiom is the restriction of the vNM independence axiom to the case where both P and R are consequences (rather than lotteries), and this is the formulation we will use. However, it should be emphasized that this axiom is not introduced as a weakening of the independence axiom but rather as a new one. As opposed to the previous axiom, the justification and appeal of this one need not stem from those of the original vNM axiom. In short, it should be examined independently of the independence axiom.

1.4. *The Results*

The main result of this paper is, roughly speaking, the following: a preference order $\geqslant$ satisfies the axioms described above if and only if there are a utility u and an increasing $\psi: \mathbb{R}^2 \to \mathbb{R}$ such that the functional

$$F(P) = \psi(E(u(P)), u(w(P)))$$

represents $\geqslant$.

A precise formulation and a proof of this result for the case of a finite set of consequences are found in Sections 2 and 3.

The case of an infinite set of consequences is dealt with separately in Section 4. It is easy to see that the main result is no longer true when the set of consequences is not assumed to be finite: consider the Maxmin-EU lexicographic order, i.e., the relation $\geqslant$ defined by

$$P \geqslant Q \text{ iff } w(P) > w(Q) \quad \text{or} \quad [w(P) \sim w(Q) \text{ and } E(u(P)) \geqslant E(u(Q))].$$

If the set of consequences is uncountable, this lexicographic order, which satisfies the axioms, is not representable by a functional as above. The general result, which is stated and proved in Section 4, shows that the axioms mentioned above are equivalent to a lexicographic decision rule, the primary criterion of which is based solely upon the Maxmin criterion, and the secondary criterion is the EU-Maxmin combination as in the finite case.

The function ψ in the representation discussed above is proven to be continuously increasing with respect to (w.r.t.) its first argument (the EU-argument) and nondecreasing w.r.t. the second (the Maxmin argument). It is, therefore, not very restrictive. However, our intuitive arguments allow us to use another version of the vNM independence axiom, which will further restrict the set of possible functions ψ: consider the case in which $w(Q)$ is preferred to both $w(P)$ and $w(R)$. If this happens to be the case, the Q mixtures do not affect the worst consequences of P and R, so that it may be assumed that the preference between P and R is not reversed when they are both mixed with Q (with the same probabilities). Using this additional axiom, we may conclude that

$$P \geqslant Q \text{ iff } f(u(w(P))) > f(u(w(Q)))$$

or

$$[f(u(w(P))) = f(u(w(Q))) \text{ and } E(u(P)) \geqslant E(u(Q))]$$

for some nondecreasing $f: \mathbb{R} \to \mathbb{R}$.

The precise formulation and the proof (both for the general case and for the finite consequences set case) are contained in Section 5.

1.5. *The Relation to Nonadditive* EU *Theory*

A special case of the decision rules axiomatized in Sections 3 and 4 is the case of a linear function ψ, that is, preferences which are represented by the functional

$$F_\alpha(P) = E(u(P)) + \alpha u(w(P)), \qquad \text{for some} \quad \alpha \geqslant 0.$$

This decision rule also happens to be a special case of the nonadditive Expected Utility models of Schmeidler (1982, 1984, 1986) and Gilboa (1987), where preferences are representable by the Choquet integral:

$$\int u\, dv = \int_0^\infty v(u \geqslant t)\, dt - \int_{-\infty}^0 [1 - v(u \geqslant t)]\, dt.$$

In these models, u is the utility and v is the probability measure, which is not necessarily additive. (It is, however, assumed to be monotone w.r.t. set inclusion.) Define

$$f_\alpha(x) = \begin{cases} (1+\alpha)^{-1} x, & 0 \leqslant x < 1 \\ 1, & x = 1. \end{cases}$$

For each $\alpha \geqslant 0$ we may consider $v_\alpha = f_\alpha(\text{Prob})$ where "Prob" is the probability measure (on the set of states of the world) underlying the lotteries under discussion. Then we obtain, for any lottery P,

$$\int u(P)\, dv_\alpha = (1+\alpha)^{-1} F_\alpha(P).$$

Hence the nonadditive EU theory and the EU-Maxmin theory coincide in this example.

To conclude we note that this example also exhausts the intersection of the two theories; that is, if a preference order satisfying all the EU-Maxmin axioms is also representable by the functional

$$\int u(P)\, df(\text{Prob})$$

for some "distortion" function f, there exists an $\alpha \geqslant 0$ such that F_α also represents it.

2. Framework and Axioms

Let X be a nonempty set of *consequences*. For $Y \subset X$, the set of *lotteries* on Y is

$$L(Y) = \left\{ P\colon Y \to [0, 1] \;\middle|\; \sum_{x \in Y} P(x) = 1,\ P \text{ has a finite support} \right\}.$$

$L(X)$ will also be denoted by L.

If $P \in L$ satisfies $P(x) = 1$ for some $x \in X$, P will be called a *simple lottery*, and when no confusion may arise, it will also be denoted by x. The set L is endowed with linear operations which are defined pointwise, and it thus constitutes a mixture set. For a given weak order $\geqslant$ on L, we may define a function $w_{\geqslant}: L \to X$ such that $w_{\geqslant}(P)$ is the $\geqslant$-worst possible consequence of P; i.e., $P(w_{\geqslant}(P)) > 0$ and for all $x \in X$, if $P(x) > 0$, then $w_{\geqslant}(P) \leqslant x$. When no confusion will be likely to arise, the subscript "$\geqslant$" will be omitted and "w" will be used alone.

For a binary relation $\geqslant \subseteq L \times L$ we consider the following axioms ($>$ and $\sim$ are defined, as usual, by $\sim\ =\ \geqslant \cap \leqslant$; $>\ =\ \geqslant \backslash \sim$):

A1. $\geqslant$ is a weak order.

A2 (weak continuity). Suppose $P < Q < R$ and $w(P) \sim w(R)$. Then there are $\alpha, \beta \in (0, 1)$ for which

$$\alpha P + (1 - \alpha) R < Q < \beta P + (1 - \beta) R.$$

A3 (weak independence). Suppose that $P, R \in L$ satisfy either

(i) $w(P) \sim w(R)$, or

(ii) P and R are simple lotteries.

Then for all $Q \in L$ and $\alpha \in (0, 1)$,

$$P \geqslant R \Leftrightarrow \alpha P + (1 - \alpha) Q \geqslant \alpha R + (1 - \alpha) Q.$$

Axiom A3 may be considered two distinct axioms, the first assuming (i) and the second assuming (ii), while both share the "then" clause. When applying the axiom we will specify which of the two axioms—to be denoted A3(i) and A3(ii), respectively—is being used.

3. The Main Result

The main result is the following:

3.1. Theorem. *Let $\geqslant$ be a binary relation on L and assume X to be finite. Then $\geqslant$ satisfies* A1–A3 *iff there exist $u: X \to \mathbb{R}$ and $\psi: \mathbb{R}^2 \to \mathbb{R}$ such that ψ is increasing and continuous with respect to the first argument and nondecreasing with respect to the second, and*

$$P \geqslant Q \Leftrightarrow \psi(E(u(P)), u(w(P))) \geqslant \psi(E(u(Q)), u(w(Q))) \qquad \textit{for all} \quad P, Q \in L.$$

(As regards the uniqueness of u and ψ, see Remark 3.6, below.)

The rest of this section is devoted to the proof of this theorem. We first introduce two new notations:

$$\text{For } x \in X, \qquad B(x) = \{y \in X \mid y \geqslant x)$$

and

$$W(x) = \{P \in L \mid w(P) \sim x\}.$$

In the sequel we will assume A1–A3 to hold, unless otherwise stated. We may now formulate

3.2. Proposition. *For each $x \in X$ there exists a function $u_x: B(x) \to \mathbb{R}$ such that*

$$P \geqslant Q \Leftrightarrow E(u_x(P)) \geqslant E(u_x(Q)), \qquad \forall P, Q \in W(x).$$

Proof. Define $\geqslant'$ on $L(B(x))$ as follows: for $P, Q \in L(B(x))$, $P \geqslant' Q$ iff for some (by A3(i) for all) $\alpha \in (0, 1)$, $\alpha x + (1-\alpha)P \geqslant \alpha x + (1-\alpha)Q$. It is obvious that $\geqslant'$ is a weak order on $L(B(x))$. Simple computations may show that $\geqslant'$ on $L(B(x))$ satisfies:

Continuity: If $P >' Q >' R$, then there are $\alpha, \beta \in (0, 1)$ for which $\alpha P + (1-\alpha)R >' Q >' \beta P + (1-\beta)R$, and

Independence: For all P, Q, $R \in L(B(x))$ and $\alpha \in (0, 1)$, $P \geqslant' R \Leftrightarrow \alpha P + (1-\alpha)Q \geqslant' \alpha R + (1-\alpha)Q$.

(Continuity is implied by A2 and A3(i), and independence by A3(i).) The desired conclusion is obtained by applying the von-Neuman–Morgenstern theorem. (See, for instance, Fishburn (1970, Theorem 8.4, p. 112)). ∎

We now wish to compare the different utilities $\{u_x\}_x$. A preliminary result is:

3.3. Lemma. *Let $x > y$, and let u_x and u_y be the corresponding utilities provided by Proposition* 3.2. *Then there are $a > 0$ and $b \in R$ for which*

$$u_y(x) = au_x(z) + b \qquad \textit{for} \quad z \in B(x).$$

Proof. The lemma is trivial if there does not exist $z \in X$ such that $z > x$. Assume, then, that there exists $\bar{z} > x$. Furthermore, without loss of generality, we assume that $u_x(x) = 0$, $u_x(\bar{z}) = 1$. Choose $b = u_y(x)$, $a = u_y(\bar{z}) - u_y(x)$. It is easy to see that Proposition 3.2 and A3(i) imply the desired result. ∎

3.4. PROPOSITION. *There exists a utility $u: X \to \mathbb{R}$ such that $P \geqslant Q \Leftrightarrow E(u(P)) \geqslant E(u(Q))$ for all $P, Q \in W(x)$ and all $x \in X$.*

Proof. Let x by a $\geqslant$-minimal consequence of X. u_x is the required function by 3.3. ∎

3.5. PROPOSITION. *There exists $\psi: \mathbb{R}^2 \to \mathbb{R}$ which is continuously increasing w.r.t. its first argument and nondecreasing w.r.t. the second, such that*

$$\psi(E(u(P)), u(w(P))) \geqslant \psi(E(u(Q)), u(w(Q))) \Leftrightarrow P \geqslant Q \quad \text{for} \quad P, Q \in L,$$

where u is the function provided by 3.4.

Proof. Let $X = \{x_1, x_2, ..., x_n\}$ where $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. We will prove by induction that for each $l \leqslant n$ there exists $\psi_l: \mathbb{R} \times \{u(x_i)\}_{i \leqslant l} \to \mathbb{R}$ as required above where L is replaced by $\bigcup_{i \leqslant l} W(x_i)$. For $l = 1$ the existence of ψ_l is trivial: $\psi(a, b) = a$ will do. Assume, then, that the assertion is true for $l - 1 \geqslant 1$. Without loss of generality we may assume that ψ_{l-1} is bounded from above. Denote

$$\underline{W} = \{P \in W(x_l) \mid \exists i < l, \exists Q \in W(x_i), Q \sim P\}$$

$$\bar{\psi} = \sup_{\substack{Q \in W(x_i) \\ i < l}} \psi_{l-1}(E(u(Q)), u(x_i))$$

$$\bar{u} = \sup_{P \in \underline{W}} E(u(P)).$$

We now turn to define ψ_l: for $i < l$, let $\psi_l(\cdot, u(x_i)) = \psi_{l-1}(\cdot, u(x_i))$. For $P \in \underline{W}$, let $Q \in W(x_{l-1})$ satisfy $Q \sim P$ and define

$$\psi_l(E(u(P)), u(x_l)) = \psi_{l-1}(E(u(Q)), u(x_{l-1})).$$

(Note that for $P \in \underline{W}$, A3(ii) and A2 guarantee the existence of such a lottery Q; i.e., if there is $Q \in W(x_i)$, $i < l - 1$ such that $Q \sim P$, there exists also $Q' \in W(x_{l-1})$ which satisfies this requirement.) If $\underline{W} = W(x_l)$, the construction of ψ_l is completed. Assume, then, that $W(x_l) \backslash \underline{W} \neq \varnothing$. Note that $W(x_l) \backslash \underline{W} = \{P \in W(x_l) \mid E(u(P)) \geqslant \bar{u}\}$, and, in particular, $\bar{u} < \infty$. By the boundedness of ψ_{l-1}, $\bar{\psi} < \infty$, and one may define

$$\psi_l(E(u(P)), u(x_l)) = E(u(P)) - \bar{u} + \bar{\psi}, \qquad \text{for} \quad P \in W(x_l) \backslash \underline{W}.$$

It is easy to see that ψ_l satisfies the required conditions, and the construction of ψ as an extension ψ_n is trivial. ∎

Proposition 3.5 proves the "only if" part of Theorem 3.1. The "if" part may be easily verified, and the theorem may be considered proved.

3.6. *Remark.* It is not astonishing to note that u is unique up to a positive linear transformation. The function ψ is also essentially unique, but the formal statement of this fact is rather cumbersome: basically, ψ is unique up to a continuously increasing transformation on $\{(E(u(P)), u(w(P)))\}_{P\in L} \subset \mathbb{R}^2$. (The definition of ψ outside this "relevant domain" is absolutely arbitrary. ψ was extended to all $\mathbb{R}^2$ only in order to simplify the formulation.) However, the monotonically increasing transformation need not be continuous on the whole range of ψ, but rather only on the half-open intervals $\{\psi(E(u(P)), u(x_i))\}_{i\leqslant n}$.

4. Extension to the Case of an Infinite Set of Consequences

We will now remove the restriction on the size of X. The main result of this section is:

4.1. Theorem. *Let $\geqslant$ be a binary relation on L. $\geqslant$ satisfies* A1–A3 *iff there exist $u, v: X \to \mathbb{R}$ and $\psi: \mathbb{R}^3 \to \mathbb{R}$ such that:*

(i) *$\psi(\cdot, \cdot, v)$ is continuously increasing w.r.t. its first argument and non-decreasing w.r.t. the second for all $v \in \mathbb{R}$;*

(ii) *$u(x) \geqslant u(y) \Rightarrow v(x) \geqslant v(y)$ for all $x, y \in X$;*

(iii) *For all $P, Q \in L$,*

$$P \geqslant Q \Leftrightarrow v(w(P)) > v(w(Q))$$

or

$$v(w(P)) = v(w(Q)) \equiv v$$

and

$$\psi(E(u(P)), u(w(P)), v) \geqslant \psi(E(u(Q)), u(w(Q)), v).$$

Note that by (ii), v is a nondecreasing transformation of u. In one extreme case v is constant, and the representation of $\geqslant$ is identical to that provided by Theorem 3.1 in the finite case. In the other extreme case, v is identical to u, and $\geqslant$ is the Maxmin-EU lexicographic order. It may be interesting to note that

4.2. Corollary. *If X is denumerable, Theorem* 3.1 *is correct.*

We have seen that Theorem 3.1 does not hold for an uncountable X. However, one may wonder whether Theorem 4.1 could be proved for a function ψ of only two variables (which is nondecreasing w.r.t. the second). The answer is negative, but the counter-example will be omitted on account of its length-to-interest ratio.

We proceed to the proof of Theorem 4.1. First, we note that Proposition 3.2 is still valid. The same can be said of Lemma 3.3. Proposition 3.4, however, though correct, calls for a new proof:

4.3. PROPOSITION. *There exists $u: X \to \mathbb{R}$ for which $P \geqslant Q \Leftrightarrow E(u(P)) \geqslant E(u(Q))$ for all $P, Q \in W(x)$ and all $x \in X$.*

Proof. Choose $x, z \in X$ such that $z > x$. (If there is no such pair of consequences, the result is trivial.) Lemma 3.3 assures that u_x is an extension of some linear transformation of u_y for all $y \geqslant x$. Hence, u_x satisfies the required condition for $y \geqslant x$. Now let $y < x$. Define $u_x(y)$ in such a way that $(u_x(y), u_x(x), u_x(z))$ will be a positive linear transformation of $(u_y(y), u_y(x), u_y(z))$. u_x is thus defined for all X. To see that its expectation represents $\geqslant$ on $W(y)$ for $y < x$, consider $y < y' \leqslant x$. Applying Lemma 3.3 for $y' > y$, one may deduce that $(u_x(y), u_x(y'), u_x(x), u_x(z))$ is a positive linear transformation of $(u_y(y), u_y(y'), u_y(x), u_y(z))$, and this observation completes the proof. ∎

In the sequel it will prove useful to define an undirected graph $G(X, E)$ by $E = \{\{x, y\} \mid \exists P \in W(x), Q \in W(y) \text{ such that } P \sim Q\}$. (That is, to each consequence $x \in X$, a vertex in G is attached, and the edges connect pairs of consequences x, y such that there is an "overlap" between $W(x)$ and $W(y)$.)

Next define a binary relation $\approx \subset X \times X$ as follows: $x \approx y$ iff there is a finite path in G from x to y. It is trivial that $\approx$ is an equivalence relation. Denote by $\mathscr{E}$ the set of all $\approx$-equivalence classes. For $C \in \mathscr{E}$, denote

$$L_C = \bigcup_{x \in C} W(x).$$

For $C, D \in \mathscr{E}$, define $C \gg D$ iff $P > Q$ for all $P \in L_C$ and all $Q \in L_D$.

4.4. LEMMA. *$\gg$ is a strict order on $\mathscr{E}$.*

Proof. Obviously, $\gg$ is anti-symmetric (i.e., $C \gg D \Rightarrow \neg(D \gg C)$) and transitive. All we have to show is that $\gg$ is complete, i.e., $C \neq D \Rightarrow C \gg D$ or $D \gg C$. Take $P \in L_C$. Denote

$$\bar{L}_D = \{Q \in L_D \mid Q > P\};\ \underline{L}_D = \{Q \in L_D \mid Q < P\}.$$

Note that $\bar{L}_D \cap \underline{L}_D = \varnothing$ and $\bar{L}_D \cup \underline{L}_D = L_D$. We claim that either $\bar{L}_D$ or $\underline{L}_D$ is empty. Assume the contrary.

Note that for each $x \in D$, either $W(x) \subset \underline{L}_D$ or $W(x) \subset \bar{L}_D$, for otherwise A2 would imply $C = D$. However, if for $x, y \in D$, $W(x) \subset \bar{L}_D$ and $W(y) \subset \underline{L}_D$, $\{x, y\}$ does not belong to E. For the same reason there is no finite path in G from x to y, a contradiction.

We may therefore deduce that for each $P \in L_C$ either

$$P > Q \text{ for all } Q \in L_D \qquad \text{or} \qquad P < Q \text{ for all } Q \in L_D.$$

A symmetric argument shows that there cannot be $P_1, P_2 \in L_C$ such that $P_1 > Q > P_2$ for all $Q \in L_D$, and the completeness of $\geqslant$ follows. ∎

The main idea of the proof of Theorem 4.1 is to distinguish among the different $\approx$-equivalence classes by the function v and to construct $\psi(\cdot, \cdot, v)$ for each equivalence class separately. To make the construction of ψ possible, each equivalence class should be "small" enough. The following Lemma defines the appropriate notion of "smallness" and guarantees that all elements of $\mathscr{E}$ indeed share this property.

4.5. LEMMA. *Let C be an $\approx$-equivalence class. Then there exists a denumerable set $\{x_i\}_{i \in Z}$ such that*

(i) $x_i \in C$ *for all* $i \in Z$

(ii) $x_{i+1} \geqslant x_i$ *for all* $i \in Z$

(iii) $\{x_i, x_{i+1}\} \in E$ *for all* $i \in Z$

(iv) *for each* $y \in C$ *there are* $i, j \in Z$ *such that* $x_i \leqslant y \leqslant x_j$.

Proof. Choose some $x_0 \in C$. We will prove the existence of $\{x_i\}_{i \geqslant 1}$ such that (i)–(iii) are satisfied and (iv) is satisfied for $y \geqslant x_0$. The rest of the proof may be carried out symmetrically.

For each $y \in C$ let $N(y)$ denote the length of the minimal path in G connecting x_0 to y. Denote

$$P_n = \{y \in C \mid y \geqslant x_0, N(y) = n\}.$$

Note that

(i) if $n > m$, $y \in P_n$, $z \in P_m$, then $y > z$;

(ii) if $n > m$ and $P_m = \varnothing$, then $P_n = \varnothing$.

We now distinguish between two cases:

(1) For all $n \geqslant 1$, $P_n \neq \varnothing$. For each $n \geqslant 1$ choose some $x_n \in P_n$. $\{x_n\}_{n \geqslant 1}$ satisfy requirements (i), (ii), and (iv), but not necessarily (iii). However, for each $n \geqslant 1$, there exists a finite path between x_{n-1} and x_n. W.l.o.g. we may assume that this path is increasing in $\geqslant$, and the conclusion of the lemma follows.

(2) For some $n \geqslant 1$, $P_n = \varnothing$. The proof is by induction on n. First assume $n = 1$. Consider the set $B = \{E(u(P)) \mid P \in W(x_0), \exists y \in C, y \sim P\}$. If $\sup B \in B$, there exists $y \in C$ which is $\geqslant$-maximal in $\{z \mid \{x_0, z\} \in E\}$, and we may choose $x_i = y$ for $i \geqslant 1$. If, however, $\sup B \notin B$, we may choose an $\geqslant$-increasing sequence $\{x_k\}_{k \geqslant 1}$ such that for any $\bar{z} \in \{z \mid \{x_0, z\} \in E\}$ there exists a $k \geqslant 1$ for which $x_k \geqslant \bar{z}$. In this case $\{x_k\}_{k \geqslant 1}$ will also satisfy (iii).

We now assume that for $n - 1 \geqslant 1$ the claim is proved. Let $\{y_k\}_{k \geqslant 1}$ be a sequence built as above, i.e., $\{x_0, y_k\} \in E$, $y_{k+1} \geqslant y_k$, and for all $\bar{z} \in \{z \mid \{x_0, z\} \in E\}$ there is a $k \geqslant 1$ such that $y_k \geqslant z$. Consider y_k for some $k \geqslant 1$. Using the lemma for $n-1$ and y_k instead of x_0, one may deduce the existence of a sequence $\{z_m^k\}_{m \geqslant 1}$ such that

(i) $z_{m+1}^k \geqslant z_m^k$, $\forall m \geqslant 1$;

(ii) for each $\bar{z}$ for which there exists an $(n-2)$-long path from y_k to $\bar{z}$, there is an $m \geqslant 1$ such that $z_m^k \geqslant \bar{z}$.

W.l.o.g. we may construct $\{z_m^k\}_{k,m \geqslant 1}$ in such a way that $z_m^k \geqslant z_m^{k-1}$ for all $k, m \geqslant 1$. Now define $x_k = z_k^k$ for $k \geqslant 1$. We wish to show that $\{x_k\}_k$ satisfies (iv), and then we may proceed as in case (1). Consider some $y \in P_{n-1}$ and suppose that $(x_0, \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{n-2}, y)$ is an increasing path in G. For $\tilde{x}_1$ there exists k_1 such that $y_{k_1} \geqslant \tilde{x}_1$. Since there is an $(n-2)$-long path between y_{k_1} and y, there exists $k_2 \geqslant k_1$ for which $z_{k_2}^{k_1} \geqslant y$. Hence $x_{k_2} = z_{k_2}^{k_2} \geqslant y$ and the lemma is proved. ∎

We are now equipped to prove

4.6. PROPOSITION. *Let C be an $\approx$-equivalence class. Then there exists $\psi: \mathbb{R}^2 \to \mathbb{R}$ which is continuously increasing w.r.t. its first argument and nondecreasing w.r.t. the second, such that*

$$P \geqslant Q \Leftrightarrow \psi(E(u(P)), u(w(P))) \geqslant \psi(E(u(Q)), u(w(Q))), \qquad \forall P, Q \in L_C.$$

Proof. Let $\{x_i\}_{i \in Z}$ be the chain provided by Lemma 4.5. It is easy to adapt the proof of Proposition 3.5 in order to prove the existence of $\psi: \mathbb{R} \times \{u(x_i)\}_{i \in Z} \to \mathbb{R}$ such that

(i) ψ is continuously increasing w.r.t. the first argument and nondecreasing w.r.t. the second;

(ii) $P \geqslant Q \Leftrightarrow \psi(E(u(P)), u(w(P))) \geqslant \psi(E(u(Q)), u(w(Q)))$ for all $P, Q \in \bigcup_{i \in Z} W(x_i)$.

For $y \in C \setminus \{x_i\}_i$ there exists an $i \in Z$ such that $x_i \leqslant y \leqslant x_{i+1}$ and $\{x_i, x_{i+1}\} \in E$. Hence for each $P \in W(y)$ there is $x \in \{x_i, x_{i+1}\}$ and $Q \in W(x)$ such that $P \sim Q$. We therefore define $\psi(E(u(P)), u(y)) = \psi(E(u(Q)), u(x))$. It is easy to see that ψ satisfies the required conditions, and it may be extended to all of $\mathbb{R}^2$ if need arises, without violating them. This completes the proof. ∎

4.7. PROPOSITION. *There are $v: X \to \mathbb{R}$ and $\psi: \mathbb{R}^2 \to \mathbb{R}$ such that*

(i) *$\psi(\cdot, \cdot, v)$ is continuously increasing w.r.t. its first argument and non-decreasing w.r.t. the second for all $v \in R$;*

(ii) *$u(x) \geqslant u(y) \Rightarrow v(x) \geqslant v(y)$ for all $x, y \in X$;*

(iii) *for all $P, Q \in L$,*

$$P \geqslant Q \Leftrightarrow v(w(P)) > v(w(Q))$$

or

$$v(w(P)) = v(w(Q)) \equiv v$$

and

$$\psi(E(u(P)), u(w(P)), v) \geqslant \psi(E(u(Q)), u(w(Q)), v).$$

Proof. For each $C \in \mathscr{E}$ choose $x \in C$ and set $v(y) = u(x)$ for all $y \in C$. Next define $\psi(\cdot, \cdot, u(x))$ to equal the function $\psi(\cdot, \cdot)$ provided by Proposition 4.6 for the $\approx$-equivalence class C. If the range of v is not all $\mathbb{R}$, extend $\psi(\cdot, \cdot, \cdot)$ in such a way that (i) is preserved. It is easily verifiable that ψ and v satisfy (i)–(iii). ▌

As the "if" part of Theorem 4.1 is trivial, the proof of the theorem is complete.

4.8. *Remark.* u is unique up to a positive linear transformation. For a given v, ψ will also be essentially unique: for every $\bar{v}$ in the range of v, $\psi(\cdot, \cdot, \bar{v})$ is unique up to a continuously increasing transformation on the "actual domain" of ψ:

$$\{(E(u(P)), u(w(P)))\}_{\substack{P \in W(x) \\ v(x) = \bar{v}}}.$$

However, v is not unique. Consider, for example, the Maxmin-EU lexicographic order for a denumerable X. It may be represented by many pairs (ψ, v) including the following extreme cases:

(i) $v = u$; $\psi(a, b, c) = a$;

(ii) $v = \text{const}$; $\psi(a, b, \text{const})$ represents the lexicographic order among ordered pairs (b, a).

5. A Lexicographically Separable ψ

The main result of this section is easier to prove in Section 4 terms than in those of Section 3 (even for the finite consequence set case), and we will borrow all the previous section's definitions and notations. We will consider a new axiom, which is yet another weakening of the vNM independence axiom:

A4. If $w(Q) \geqslant w(P), w(R)$, the for all $\alpha \in (0, 1)$, $P \geqslant R \Leftrightarrow \alpha P + (1-\alpha)Q \geqslant \alpha R + (1-\alpha)Q$.

Now we have

5.1. Theorem. *A binary relation $\geqslant$ on L satisfies* A1–A4 *iff there are u, v, ψ as specified in Theorem* 4.1 *such that*

$$\psi(a, b, c) = a, \qquad \forall (a, b, c) \in \mathbb{R}^3.$$

Note that the formulation is equivalent to the one appearing in Subsection 1.4.

Turning to the proof, we begin with the "only if" part and assume A1–A4 to hold. The main observation is:

5.2. LEMMA. *Suppose* $x > y$, $P_1, P_2 \in W(x)$, $Q_1, Q_2 \in W(y)$, $P_1 \sim Q_1$, *and* $E(u(P_2)) - E(u(P_1)) = E(u(Q_2)) - E(u(Q_1))$. *Then* $P_2 \sim Q_2$.

Proof. First assume $P_2 \geqslant P_1$ (whence $Q_2 \geqslant Q_1$). Let $z \in X$ satisfy $u(z) > E(u(Q_2))$. (There must exist such a consequence z since Q_2 assigns a positive probability to y.) As $E(u(Q_2)) \geqslant E(u(Q_1)) \geqslant E(u(P_1)) \geqslant u(x)$, we obtain $u(z) > u(x)$. Now let $\alpha \in (0, 1)$ satisfy

$$\alpha z + (1 - \alpha) P_1 \sim P_2.$$

Using the expected utility representation on $W(x)$ and on $W(y)$, one obtains

$$\alpha z + (1 - \alpha) Q_1 \sim Q_2.$$

However, A4 implies $\alpha z + (1 - \alpha) P_1 \sim \alpha z + (1 - \alpha) Q_1$, and the result follows.

Now assume $P_2 < P_1$. If $P_2 > Q_2$, let Q_3 be a mixture of Q_2 and Q_1 such that $P_2 \sim Q_3$. Let P_4 be a mixture of P_1 and P_2 such that

$$E(u(P_4)) - E(u(P_2)) = E(u(Q_1)) - E(u(Q_3)).$$

By the first part of the lemma, $P_4 \sim Q_1 \sim P_1$, a contradiction. A symmetric construction shows that $P_2 < Q_2$ is also impossible, and the lemma is proved. ∎

5.3. PROPOSITION. *Let C be an $\approx$-equivalence class. There exists a nondecreasing $f_C \colon \mathbb{R} \to \mathbb{R}$ such that*

$$P \geqslant Q \Leftrightarrow E(u(P)) + f_C(u(w(P))) \geqslant E(u(Q)) + f_C(u(w(Q))), \qquad \forall P, Q \in L_C.$$

Proof. We distingish between the cases of bounded and unbounded u:

Case 1. $\sup u = \infty$.

Let $\{x_i\}_{i \in Z}$ be the chain provided by Lemma 4.5. We first wish to show that $\{x, y\} \in E$ for all $x, y \in C$. It suffices to show that $\{x_0, x_i\} \in E$ for all $i \in Z$, and that will be done by induction on $|i|$. For $|i| = 1$ our claim follows from the definition of $\{x_i\}_{i \in Z}$. Assume, then, that it is true for $|j| \leqslant i - 1$. Suppose $P_1, P_2 \in W(x_{i-1})$, $Q_1 \in W(x_0)$, $R_2 \in W(x_i)$, and $P_1 \sim Q_1$, $R_2 \sim P_2$. Let $Q_2 \in W(x_0)$ satisfy $E(u(Q_2)) - E(u(Q_1)) = E(u(P_2)) - E(u(P_1))$. By the previous lemma, $R_2 \sim P_2 \sim Q_2$. That is, $\{x_0, x_i\} \in E$. Shifting all indices by i will also yield $(x_0, x_{-i}) \in E$. Now fix $x \in C$, and for each $y \in C$ define

$$f_C(u(y)) = E(u(Q)) - E(u(P))$$

for some $Q \in W(x)$, $P \in W(y)$ such that $Q \sim P$. Note that Lemma 5.2 assures that f_C

is well defined. To see that f_C thus defined indeed satisfies our condition, let $y, z \in X$ and $P \in W(y)$, $R \in W(z)$. Let $t \in X$, $\alpha \in (0, 1)$, and $Q_y, Q_z \in W(x)$ satisfy

$$\tilde{P} \equiv \alpha t + (1-\alpha) P \sim Q_y$$

$$\tilde{R} \equiv \alpha t + (1-\alpha) R \sim Q_z$$

$$t \geqslant y, z.$$

By A4, $P \geqslant R \Leftrightarrow \tilde{P} \geqslant \tilde{R}$. However,

$$\begin{aligned} \tilde{P} \geqslant \tilde{R} \Leftrightarrow Q_y \geqslant Q_z &\Leftrightarrow E(u(Q_y)) \geqslant E(u(Q_z)) \\ &\Leftrightarrow E(u(\tilde{P})) + f_C(u(y)) \geqslant E(u(\tilde{R})) + f_C(u(z)) \\ &\Leftrightarrow E(u(P)) + f_C(u(y)) \geqslant E(u(R)) + f_C(u(z)). \end{aligned}$$

Case 2. $\bar{u} \equiv \sup u < \infty$.

This case may be easily reduced to the first one as follows: first extend X to include $[\bar{u}, \infty]$; next redefine ψ on $\{(u, \cdot, \cdot) \mid u > \bar{u}\}$ (where it was defined quite arbitrarily in Theorem 4.1) so that

$$\psi(a, b, c) = \psi(\bar{a}, \bar{b}, c)$$

implies

$$\psi(a+d, b, c) = \psi(\bar{a}+d, \bar{b}, c) \qquad \text{for all} \quad d \geqslant 0.$$

Finally, extend $\geqslant$ in accordance with ψ, and use the first part of the proof. ∎

5.4. Lemma. *If $P, Q \in L_C$ for some $C \in \mathscr{E}$ satisfy*

$$f_C(u(w(P))) > f_C(u(w(Q))),$$

then $P > Q$.

Proof. Assume the contrary, i.e., $P \leqslant Q$. Choose $R \in L$ such that $w(R) \geqslant w(P), w(Q)$. Denote

$$P_\alpha = \alpha R + (1-\alpha) P$$

$$Q_\alpha = \alpha R + (1-\alpha) Q$$

for $\alpha \in (0, 1)$. By A4, $P_\alpha \leqslant Q_\alpha$ for all α. However, for α sufficienly close to 1,

$$E(u(P_\alpha)) + f_C(u(w(P_\alpha))) > E(u(Q_\alpha)) + f_C(u(w(Q_\alpha)))$$

which is a contradiction by 5.3. ∎

5.5. Proposition. *The functions v and ψ may be chosen in such a way that f_C is constant for each $C \in \mathscr{E}$.*

Proof. Trivial in view of 5.4. ▮

Proposition 5.5 proves the "only if" part of Theorem 5.1. Its "if" part is trivial, so the proof is complete.

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