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Aggregation of semiorders: intransitive indifference makes a difference*

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Summary. A semiorder can be thought of as a binary relation P for which there is a utility u representing it in the following sense: xPy iff $u(x) - u(y) > 1$. We argue that weak orders (for which indifference is transitive) can not be considered a successful approximation of semiorders; for instance, a utility function representing a semiorder in the manner mentioned above is almost unique, i.e. cardinal and not only ordinal. In this paper we deal with semiorders on a product space and their relation to given semiorders on the original spaces. Following the intuition of Rubinstein we find surprising results: with the appropriate framework, it turns out that a Savage-type expected utility requires significantly weaker axioms than it does in the context of weak orders.

1. Introduction

1.1 Background

The bulk of economic theory literature assumes that a decision maker's preference relation P may be represented by a utility function $u(\cdot)$ as follows: xPy iff $u(x) - u(y) > 0$.

The axioms P must satisfy for such a representation, as well as the very existence of a binary preference relation, have all been criticized over the years on both theoretical and empirical grounds. Yet, most of the axioms, including transitivity of P, are still widely accepted.

One exception is the transitivity of indifference: a representation as above, trying to retain this property, implies that the decision maker strictly prefers x to y whenever $u(x)$ is even slightly larger than $u(y)$. Most theorists would probably not take this axiom literally. To use Luce's (1956) classical example, it seems unlikely that a person will be able to distinguish between a cup of coffee with 2.534 grains of

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sugar in it and another with 2,533 grains. Regardless of one's taste for sugar, one can hardly express strict preference between these two alternatives. Similarly, the most preferred stream of consumption an economic agent can afford with an annual income of $30,000.00 is most likely to be indistinguishable from the corresponding consumption affordable at $30,000.01. Indeed, if asked to compare between the numbers 30,000.00 and 30,000.01 in their decimal representation, people are likely to notice the difference in the last digit. However, the consumer's preference order is typically defined over commodity bundles. Considering physical bundles, as opposed to symbolic representation thereof, a strict preference for the $30,000.01-bundle to the $30,000.00 one appears to be a mathematical idealization of the everyday notion of "preferences".

An obvious alternative is to claim that economic agents are bound to be indifferent between indistinguishable bundles. Thus, a cup of coffee with n grains of sugar is just as preferred to one with $(n+1)$ grains, and the "best" bundle worth n cents is equivalent to that corresponding to $(n+1)$ cents. This being the case for all $n \geq 0$, transitivity of indifference implies that the consumer does not care about the amount of sugar in the coffee at all, is indifferent among the "best" bundles for any two income levels, and so forth. Needless to say, such a model is neither realistic nor useful.

Moreover, psychology advances the Weber–Fechner Law (Weber, 1834; Fechner, 1860) which says, roughly, that people cannot discern between very close objects, and only when the difference in mass, temperature, length, pressure, and so forth exceeds a certain "just noticeable difference" does a distinction emerge in their minds. Weber–Fechner's Law also says, that for each scale – mass, for instance, – there is a context-dependent constant $\lambda > 1$ such that the individual will notice a difference between two magnitudes (with some fixed probability p at least) only if their ratio exceeds λ (or drops below $1/\lambda$). On the appropriate logarithmic scale, we therefore get the representation

$$xPy \quad \text{iff } u(x) - u(y) > 1. \tag{1}$$

Semiorders, defined by Luce (1956), are transitive binary relations whose defined indifference relations may not be transitive. For the sake of discussion one may take (1) as a definition of a semiorder although it is equivalent to Luce's definition only if the set of alternatives is finite. (The precise definition is given below.) For discussion of representations of semiorders – and the more general class of interval orders – see Fishburn (1970, 1985), Manders (1981), Gensemer (1987), Bridges (1983), Chateauneuf (1987), Beja and Gilboa (1992) and Fishburn and Nakamura (1991)[1].

From the viewpoint of economic theory it is quite reasonable to argue, then, that in "reality" people have semiorders or even less structured preferences in which

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[1] Another interpretation of semiorders may be the uncertainty about a measured quantity, which is due to measurement errors. According to this interpretation, absence of preference may result from the possibility that the "true" underlying magnitudes are in reversed order as compared to the measured ones.

For an application in which semiorders arise in this interpretation see Lapson and Lugachev (1983).

indifference is intransitive. But weak orders can still be taken as an approximation, or a mathematical idealization, of reality; an idealization that simplifies matters just like the continuum simplifies the representation of very large discrete sets.

The main point of this paper is that this idealization is far from innocuous. Surprisingly enough, we have good news: some aspects of economic theory are conceptually simpler with semiorders than with weak orders. Although weak orders simplify the mathematics, they require considerably stronger axioms for results such as derivation of the expected utility representations.

The following example, noted in Beja and Gilboa (1992), illustrates the point. Suppose P is a semiorder with representation (1) and u is onto $\mathscr{R}$. Then u is almost unique. Should v also represent P, there will be a function f such that $v \equiv f(u)$, but f cannot by *any* monotone transformation. It can be defined arbitrarily up to monotonicity preservation on $[0, 1)$ into itself, but then it must satisfy $f(x + \mathrm{n}) = f(x) + n$ for all $x \in [0, 1)$ and $n \in Z$.

If we think of the just noticeable difference 1 as relatively small, then u is almost unique. It thus makes sense to discuss properties such as concavity or convexity of the utility function, properties that the mathematical idealization of weak orders renders meaningless.

Thus, the ranking of differences implied by "larger than the just-noticeable-difference" versus "not larger than the just-noticeable-difference" is naturally given in the original preferences, and it suffices for fixing u almost uniquely without the additional assumption that decision makers can answer questions like "do you prefer x to y more than you prefer w to z ?" in a meaningful and coherent way.

In a similar way, this paper shows that when semiorders are primitive, one needs relatively weak axioms to derive expected utility (or weighted average) representations for preference relations over a product space of given spaces – for instance, the space of acts in decision under uncertainty.

1.2 Decisions under uncertainty as aggregation of semiorders

In the context of uncertainty the objects of choice ("acts") are functions from states of nature to outcomes. If we assume the state space to be finite, acts are simply vectors of outcomes, indexed by the states.

We will assume that, should state $1 \leq i \leq n$ occur, the decision maker has semiordered preferences P_i on some space of outcomes X_i. (We will later set all these to be identical, but some of our results do not require this additional restriction.) The question we would like to address is, what semiorders P on the product space $X = \Pi X_i$ are "compatible" with the given semiorders P_i?

In search for an answer we introduce two properties: monotonicity and consistency. Monotonicity is a Pareto dominance axiom: (appropriately defined) weak preference in all components and strict preference in at least one of them implies preference between vectors under discussion. (See section 2 below for details.) Consistency is a very weak version of the "sure thing principle", namely, that fixing (n-1) components of a vector x and restricting P to the alternatives which differ on the i-th component alone, one gets the corresponding P_i. If the P_i's were weak orders, this requirement would boil down to our monotonicity assumption.

Thus, the assumptions of monotonicity and consistency have almost no bite in the context of weak orders; they allow for any monotone aggregation of the primitive orders, and say practically nothing about the tradeoffs between the various spaces X_i. To be precise, if u_i represents a weak order P_i on X_i and $f:\mathscr{R}^n \to \mathscr{R}$ is monotone, then the weak order P on X defined by $f(u_1, \ldots, u_n)$ is monotone and consistent with $(P_i)_i$.

By contrast, when the P_i's are semiorders, we prove that – under some mild technical assumptions – there is an almost unique semiorder P on X which is both monotone and consistent with the given P_i's, and that each such P has an almost unique representation.

Moreover, if u_i represents P_i with a just-noticeable-difference of 1, one of the monotone and consistent semiorders will be represented by Σu_i, namely

$$xPy \text{ iff } \Sigma u_i(x_i) - \Sigma u_i(y_i) > 1.$$

(see Theorems A and B in section 3.)

In the context of uncertainty, all spaces X_i are identical – namely, the set of outcomes, – where a vector $(x_1, \ldots, x_n)$ denotes an act, i.e. a function from states of the world to outcomes. If all P_i's are also identical, we obtain the representation $\Sigma u(x_i)$. Thus, our axioms lend support to the Laplace "Insufficient Reason" criterion, which is a special case of expected utility maximization, where all states of nature are considered equally likely.

Note that the Laplace criterion is derived from monotonicity and consistency alone. Comparing these axioms to the main axioms of Savage (1954), the consistency corresponds to his axiom P2 – "the Sure-Thing principle" – restricted to singleton events, while monotonicity corresponds to axiom P3 – "stochastic dominance", or "monotonicity". There is no need to require the full strength of P2, nor axiom P4 – "comparability of events", or "separation of tastes and beliefs" – and yet one obtains a stronger result, namely an expected utility representation in which all states of nature are equiprobable.

However, the Laplace criterion, ascribing equal probabilities to all states, is quite restrictive and may point to a flaw in our assumptions: indeed, assuming all X_i's are the same is intrinsic to the problem of decision making under uncertainty, but the assumption that all P_i's are also identical may be too strong. Instead, we could allow the decision maker's sensitivity to differences between alternatives to depend on the likelihood of the associated state of nature. That is, all alternatives are ordered on some utility scale by a single function u. However, the just-noticeable-difference of the semiorder P_i may vary with i, depending on how likely the decision maker finds state i.

We are therefore interested in the following question: given several semiorders P_i on the same space X, when is there a single utility function u from X onto $\mathscr{R}$ and constants $\delta_i > 0$ corresponding to P_i, such that

$$xP_iy \text{ iff } u(x) - u(y) > \delta_i \text{ ?}$$

In this paper we restrict our attention to the case of all δ_i being rational, for which we provide a complete axiomatization (Theorem C below). Using this result and the previous ones we offer an axiomatization of a semiorder P on X^n represented by

$$xPy \quad \text{iff } \Sigma p_i u(x_i) - \Sigma p_i u(y_i) > 1$$

for rational probabilities p_i (Corollary D).

We thus obtain an alternative derivation of expected utility representation of preferences. Furthermore, this derivation implicitly provides a new behavioral definition of subjective probability: the probability of a state is the inverse of the associated just-noticeable-difference. In particular, this behavioral definition captures the intuition that the more sensitive is the decision maker to differences in a given state, the higher is the "subjective probability" associated with this state. One may choose a cognitive interpretation by which the more likely is a state, the more cognitive resources will be devoted to it, and the smaller is the resulting just-noticeable-difference. However, the definition only requires the behavioral data of the observed semiorders.

While certainly called for, additional discussion may prove more fruitful after a presentation of a formal model. We therefore turn now to section 2, which presents preliminary definitions and quotes some results from the literature. Section 3 contains the results, while section 4 continues the discussion. Finally, proofs and related analysis are provided in section 5.

2. Preliminaries and basic definitions

The central issue of this paper is semiorders. The formal definition is the following:

A binary relation P on X is a semiorder (Luce (1956)) if for all x, y, z, w in X

1) not xPx (P is irreflexive);
2) if xPy and zPw then xPw or zPy;
3) xPy and yPz then xPw or wPz.

For a given semiorder P define binary relations I, Q, E and Q^0 on X as follows:

xIy if not xPy and not yPx;
xQy if $\exists z$ in X such that either 1) xPz and not yPz
or 2) zPy and not zPx;

xEy if not xQy and not yQx;
xQ^0y if xQy or xEy

These relations appear in the literature under many names. Here we follow Beja and Gilboa (1992) most closely.

Any scripts on P will be carried over to its associated I, Q, E and Q^0. It is well known that if P is a semiorder, then the associated Q is a weak order, i.e. for all x, y, z in X, Q satisfies

1) not xQx (Q is irreflexive);
2) if xQy and yQz then xQz (Q is transitive);
3) if xQy then xQz or zQy.

Scott and Suppes (1958) proved that if X is finite, then there exists a utility function on X such that

$$\text{for any } x,y \text{ in } X \qquad xPy \quad \text{iff } u(x) > u(y) + 1 \quad \text{and} \quad xQy \quad \text{iff } u(x) > u(y).$$

Manders (1981) and Beja and Gilboa (1992) showed that for this result to be true for a countable X an additional axiom is needed saying that for every x in X and every infinite sequence $x_1, x_2, \ldots$ in X if $x_i P x_{i+1}$ for $i = 1, 2, \ldots$ then for some n xPx_n and if $x_{i+1} P x_i$ for $i = 1, 2, \ldots$ then for some n $x_n P x$. Beja and Gilboa (1992) provide characterization of the just-noticeable-difference representation for a general (not necessarily countable) X. To avoid technical difficulties we will generally assume in this paper that range$(u) = \mathscr{R}$.

Let us recall the standard definition of composition of binary relations on X: given two binary relations B_1 and B_2, define $B_1 B_2$ by xB_1B_2y if there exists z in X such that xB_1z and zB_2y.

Note that successive applications of this definition render composition of more than two relations well-defined. In particular, we will use the notation $B^n = BB^{n-1}$ for $n \geq 2$ (with $B^1 = B$).

Let $X_1, \ldots, X_n$ be given sets and let there be semiorders P_i defined on every X_i, $i = 1, \ldots, n$. Assume that P is a semiorder on $X = X_1 \times \cdots \times X_n$. For a generic element x in X, x_i will denote its i-th component, x_{-i} will stand for $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and (x_{-i}, y_i) for $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

Further suppose that P and Q are such that there exists a utility function u from X onto $\mathscr{R}$ such that xPy iff $u(x) > u(y) + 1$ and xQy iff $u(x) > u(x)$ for all x and y in X. We will say that u represents P and call P representable. From here on let us assume that every P_i is representable.

Note. The function u_i is uniquely defined up to a strictly increasing transformation $f(\cdot)$, satisfying $f(a) = f(a-1) + 1$ for all a in $\mathscr{R}$, where 1 is the jnd. The set of all such functions f may be identified with the set $\{f\colon [0,1) \to [0,1) | f \text{ is monotone}\}$. That is, f can be arbitrarily determined on $[0,1)$, as long as monotonicity is preserved, and it is then uniquely defined on $\mathscr{R}$ by "replication" of its values on $[0,1)$. For details, see Beja and Gilboa (1992).

Sometimes it is easier to start with u and then derive P from it, which is representable by that u. In such case we will say that P is defined by u.

Now we can define some properties P may possess w.r.t. $P_1, \ldots, P_n$:

Definition 1. P on X is *monotone* with respect to the semiorders $P_1, \ldots, P_n$ (hereafter monotone) if $\forall x, y \in X$ the following holds: if $x_i Q^0{}_i y_i$ for all $i \in N \equiv \{1, \ldots, n\}$ then xQ^0y.

(As above, Q_i and Q are the corresponding weak orders.)

Definition 2. P is *consistent* with $P_1, \ldots, P_n$ (hereafter consistent) if for all $i \in N$ and for all $x_i, y_i \in X_i$, $x_i P_i y_i$ iff $(z_{-i}, x_i) P (z_{-i}, y_i)$ for all $z_{-i} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$.

One may separate "consistency" into two requirements: first, that whenever $x_{-i} = y_{-i}$, the preference between x and y depends only on x_i and y_i. (The "Sure-Thing principle" restricted to singleton events.) And second, that this latter preference be determined by P_i (defined on X_i). Recall, however, that the observed preference relation is P. Thus, if P satisfies the restricted Sure-Thing principle, we *define* P_i to be the induced relation on X_i. Since P is assumed to be a representable semiorder, the derived P_i will also be such.

We choose to start with the semiorders P_i and P as separate for two related reasons: clarity of exposition and the alternative, cognitive interpretation, according to which the P_i's are primitives from which P is derived.

Let us define for each semiorder P on X the P-topology as follows: $x_n \to X$ if for every $y \in X$ for which yQx, there exists M such that $\forall m \geq M\ yQx_m$ and for $y \in X$ for which xQy there exists M such that $\forall m \geq M\ x_m Qy$. The continuity of P we are about to define means that if a sequence $\{x_{ik}\}$ in X_i converges to x_i in X_i in the P_i-topology (for all i), then $\{x_k\}$ converges to x in the P-topology. (In the presence of monotonicity and consistency, this is tantamount to saying that the P-topology on X is the product topology defined by the P_i-topologies on X_i, $i = 1, \ldots, n$.)

Definition 3. 1) P is *continuous from above* [*below*] with respect to (P_i) $i = 1, \ldots, n$ if for all i, for every sequence $\{x_{ik}\}$ converging to x_i in X_i and for all y in X if $(x_1, \ldots, x_n)P(y_1, \ldots, y_n)$ $[(y_1, \ldots, y_n)P(x_1, \ldots, x_n)]$ then there exists M such that for every $k > M (x_{1k}, \ldots, x_{nk})P(y_1, \ldots, y_n)[(y_1, \ldots, y_n)P(x_{1k}, \ldots, x_{nk})]$.

2) P is *continuous* with respect to $P_1, \ldots, P_n$ (hereafter continuous) if it is continuous both from above and from below.

Finally, the symbol $\neg$ will stand for negation.

3. Main results

Our main results can be reduced to three theorems.

Theorem A states existence of and characterizes monotone, consistent and continuous semiorders on a product space.

Theorem A. Let $\{P_i\}_{i \in N}$ be semiorders on $\{X_i\}_{i \in N}$ represented by $\{u_i\}_{i \in N}$ respectively. Let P be a semiorder on $X = \Pi_{i \in N} X_i$ represented by u. Then the following are equivalent:

(i) P is consistent, monotone and continuous w.r.t. $P_1, \ldots, P_n$;
(ii) There is a strictly monotone and continuous function $f_u\colon \mathscr{R}^n \to \mathscr{R}$

satisfying

$$f_u(a_{-i}, a_i) = f_u(a_{-i}, a_i + 1) - 1, \quad \forall a \in \mathscr{R}^n, \quad \forall i, \tag{2}$$

such that $u = f_u(u_1, \ldots, u_n)$.

In particular, if $u(x) = \Sigma_i u_i(x_i)$, the semiorder P on X, defined by u, satisfies (i).

Theorem B gives the notion of "almost uniqueness" of such a semiorder on the product space. It limits the extent to which two representable semiorders P_a, P_b, which are both consistent, monotone and continuous with respect to given $\{P_i\}_{i \in N}$, can disagree. While it might be the case that there are $x, y \in X$ such that $xP_a y$ and $yP_b x$ (see observation 5.5 below), it can only be true of alternatives x and y which are "close" in an appropriate sense.

Theorem B. Let $\{P_i\}_{i\in N}$ be semiorders on $\{X_i\}_{i\in N}$ represented by $\{u_i\}_{i\in N}$ respectively. Suppose that P_a and P_b are two representable semiorders on X which are both consistent, monotone and continuous with respect to $P_1, \ldots, P_n$. Then $(P_a)^n \subseteq P_b$ and $(P_b)^n \subseteq P_a$.

Theorem C deals with a joint representation of several semiorders on the same space. It will need additional axioms. For two semiorders P and P' on a certain space X define the following conditions.

A1. The compositions of P and P' is commutative, namely, $PP' = P'P$.
A2. For any $k, m \in N$ either $(P)^k \subseteq (P')^m$ or $(P')^m \subseteq (P)^k$ and for some $k, m \in N$, we also have $(P)^k = (P')^m$.

Assumptions A1 and A2 deal with the case when one uses two criteria, each expressed by a semiorder to compare alternatives. A1 is a symmetry assumption. It says that the comparison of alternatives based on sequential application of the two criteria does not depend on the order in which the criteria appear in the sequence.

A2 is a comparability axiom. Consider its first part with $k = m = 1$. Then it simply says that either P-preference always implies P'-preference or vice versa. That is, either P' is "more sensitive" than P or P is "more sensitive" than P'. Similarly, this axiom requires that any power of P and any power of P' be comparable in this sense. Finally, the second part of A2 stipulates that for some $k, m \in N$, P^m and $(P')^k$ are equally sensitive.

Theorem C. Given semiorders $P_1, \ldots, P_n$ on a set X such that P_i is represented by $u_i(\cdot)$, $i = 1, \ldots, n$, the following are equivalent:

(i) there exist a function u: $X \to \mathscr{R}$ with range $(u) = \mathscr{R}$ and positive rational numbers $\delta_1, \ldots, \delta_n$ such that for all $i = 1, \ldots, n$, and for all x, y in X

xP_iy iff $u(x) - u(y) > \delta_i$, and
xQ_iy iff $u(x) > u(y)$;

(ii) for all i and j in $\{1, \ldots, n\}$ P_i and P_j satisfy A1 and A2.

Corollary D applies the previous results to expected utility representation.

Corollary D. Let X' be a set and let $P_1, \ldots, P_n$ be semiorders on it represented by $u_1, \ldots, u_n$ (respectively), where range $(u_i) = \mathscr{R}$ for all i. Suppose that for every $i, j \in N$ P_i and P_j satisfy A1 and A2. Define P on $X = (X')^n$ by $u = \Sigma(1/\delta_i)u_i$ for δ_i obtained from Theorem C. Then

(i) P is continuous, consistent and monotone w.r.t. $P_1, \ldots, P_n$;
(ii) If P' is another semiorder on X which is continuous, consistent and monotone w.r.t. $P_1, \ldots, P_n$ then $(P')^n \subseteq P$ and $P^n \subseteq P'$.

4. Discussion

4.1 Some comments on the literature

The most closely relevant work, which motivated this study, is Rubinstein (1988). He discusses preference orders over simple lotteries. A lottery is "simple" if it promises a certain monetary prize x with probability p and zero with probability $1-p$.

Rubinstein assumes that there are two "similarity" relations, $\sim_x$ and $\sim_p$, defined on the set of monetary prizes and the set of probabilities, respectively, with the interpretation that two "similar" magnitudes (prizes or probabilities) are indistinguishable in the decision maker's mind. He then considers weak orders on simple lotteries and defines the "star" (*) property as follows: A weak order satisfies property (*) with respect to $\sim_x$ and $\sim_p$ if, whenever two lotteries are similar in one component but not the other, then the other component determines the preference between the two. Rubinstein proves that for given similarity relations there is an almost unique weak order on lotteries satisfying property (*). Contending that this property is a basic feature of any reasonable decision process, he concludes that there is a basic flaw in axiomatic theories justifying expected utility maximization because this decision rule may well be inconsistent with property (*).

Rubinstein's similarity relations are, in fact, indifference relations induced by semiorders. Our original intuition was that his results hinge on the fact that a preference in the lottery space is assumed to be a weak order, while on the more primitive spaces only semiorders are given. In this case his conclusions appear dubious since it seems unreasonable that a decision maker who cannot discern small differences between monetary prizes as such will discriminate perfectly between more complicated objects such as lotteries.

This intuition seems to have been shared by Aizpurua, Nieto and Uriarte (1988) and Aizpurua et al. (1989). They allowed preferences on lotteries to exhibit intransitive indifference but found that Rubinstein's results are robust with respect to this generalization. To cope with the "over-determination problem" they introduced "correlated similarities" – allowing the similarity relation on the probability space to depend on the associated monetary prize. Thus, they found that expected utility maximization was not inconsistent with property (*), though this result (as well as Rubinstein's) could not be extended to lotteries with more than two possible prizes.

Our approach is slightly different. We require the binary relation on the product space to be a semiorder (a stronger requirement than theirs), and find that proprety (*) is too strong; there typically will not be semiorders on the product space which satisfy property (*) with respect to the given ones on the original spaces. Moreover, this is true even for two original spaces.

Thus we find that our monotonicity and consistency requirements are more appropriate for the context of semiorders than the (*) property. As mentioned in subsection 4.1 above, our results are quite different from Rubinstein's as welll as from Aizpurua et al.'s (1988, 1989). In particular, while they criticize the expected utility paradigm, we find support for the semiordered version of it.

Our results are also related to Luce (1973), who axiomatized an additive aggregation of semiorders. However, his approach is slightly different, his result is restricted to the case of two semiorders, and we find our axioms somewhat simpler.

The problem of joint representation of several semiorders was addressed by Cozzens and Roberts (1982) and Doignon (1987). Their results are restricted to finite sets of alternatives, and the axioms they use seem to be not as primitive as ours.

4.2 *Are semiorders reasonable?*

Some economic theorists tend to reject semiorders as unreasonable on account of the discontinuity at the just noticeable difference (jnd). How likely is it, they would ask, that an individual will not distinguish between 0 and 99 grains of sugar in the famous cup of coffee, but when an additional grain is added preference suddently emerges? Thus, their argument would continue, instead of assuming that a difference of 1 is distinguishable from a difference of 0, semiorders suggest that a difference of 100 is distinguishable from one of 99.

Without any claim to originality, we would like to make three comments. While the first should settle the issue, the other two may be of methodological value.

1. The argument against semiorders presented above is based on too literal an interpretation: it is not the case that as long as the difference between two magnitudes is below the just-noticeable difference the probability of discernment is 0 and just above the just-noticeable difference it "jumps" to 1. As one would except, the probability of an individual noticing a difference is a continuously increasing function of the latter. In fact, the meaning of "noticing the difference" is defined in psychological experiments as "noticing the difference with probability α or more", where popular values of α are 0.50 and 0.75.

Thus, in a more elaborate model one may think of an individual's preferences as a family $\{P_\alpha\}_{\alpha\in[0,1]}$ of semiorders such that $P_\alpha \subseteq P_\beta$ for $\alpha \geq \beta$. (See Roberts (1971)). Hence our main point is only strengthened by this argument: a decision maker's "real" preferences contain even more information than is conveyed by one semiorder, let alone by the induced weak order. Yet we show that one semiorder suffices to obtain relatively strong results.

2. Even if we stick to the "naive" interpretation of a semiorder or, alternatively, focus on $\alpha = 1$ (i.e., certain discernment) and disregard any $\alpha < 1$, we do not find the semiorder model as absurd as suggested by its opponents. Indeed, if one is asked to determine the minimal number of grains of sugar, the presence of which would be detected (by him/her) with "practical certainty", say, probability 0.99, it is hard to focus on any individual number as the minimal one. Yet, the existence of such a minimal number cannot be refuted. Its measurement may be rather problematic, but the fact it exists (combined with Weber–Fechner's law) may still be used in the derivation of theoretical results.

3. On top of the two arguments raised above, we would like to point out a major conceptual difference between the difficulties implied by zero and positive jnd's. Assuming the indifference is transitive (i.e., a zero jnd) is attacked on the grounds that the alleged preference *stated by a decision maker* is unreasonable. By comparison, the criticism of semiorders is of second-order: it is some relationship among stated preferences that is questioned. Put differently, the comparison between the difference between 0 and 99 and that between 0 and 100 is not a comparison the decision maker is assumed to make in this model. A modeler (or an introspective decision maker) may wonder what is the minimal n such that n grains are distinguishable from 0, and, being boundedly rational, may run into the problem discussed in (2) above. Yet this problem seems to be secondary to the first one.

5. Proofs and auxiliary results

Let $X_1, \ldots, X_n$ be given sets with semiorders $P_1, \ldots, P_n$ defined on them respectively. Let $X = X_1 \times \cdots \times X_n$, and let P be a semiorder on X represented by u. We assume these conditions unless otherwise stated.

To motivate our definition of monotonicity and consistency, let us compare this to the (*) property of Rubinstein's (1988).

Definition. P satisfies the (*) property w.r.t. $(P_i)_{i \in N}$ if the following holds: for every x, $y \in X$, $\neg(y_j P_j x_j)$ for all $j \neq i$ and $x_i P_i y_i$ implies xPy.

We will show that even for the case $n=2$ the (*) property is too strict for aggregation of semiorders. In the next observation P_i are assumed to be representable, whence range $(u_i) = \mathscr{R}$. Note, however, that it suffices that range $(u_i) \supset (a, b)$ for some $b > \mathrm{a} + 1$.

Observation 5.1. If P_1 and P_2 are representable semiorders, there does not exist a representable semiorder P which satisfies the (*) property.

Proof. Let x, y, z in X be such that $u_1(y_1) = u_1(x_1) - \varepsilon + 1$, $u_2(y_2) = u_2(x_2) - \varepsilon - 1$, $u_1(z_1) = u_1(x_1) + \varepsilon + 1$, $u_2(z_2) = u_2(x_2) + \varepsilon - 1$, where ε is a positive number less than 1. It follows from the (*) property that xPy and zPx. By transitivity of P it implies zPy. Moreover, if u represents P, $u(x) - u(y) > 1$ and $u(z) - u(x) > 1$ follow from xPy and zPx, respectively. Hence, $u(z) - u(y) > 2$. Denote the interval $[(u_1(y_1), u_2(y_2)), (u_1(z_1), u_2(z_2))]$ by d. For any two points v, w in X such that $(u_1(v_1), u_2(v_2))$, $(u_1(w_1), u_2(w_2)) \in d$ and $u_1(v_1) > u_1(w_1)$ we get $u(v) - u(w) > 2$, a contradiction. //

The following lemma shows that our concepts of monotonicity and consistency are indeed weaker than strong consistency as implied by Rubinstein (1988) and they allow us to achieve some positive results.

Lemma 5.2. For all representable semiorders $P_1, \ldots, P_n$ on $X_1, \ldots, X_n$ respectively there exists a consistent, monotone and continuous representable semiorder P.

Proof. Let u_i be such that for all $x_i, y_i \in X_i$, $x_i p_i y_i$ iff $u_i(x_i) > u_i(y_i) + 1$ and $x_i Q_i y_i$ iff $u_i(x_i) > u_i(y_i)$, $i = 1, \ldots, n$.

Define $u(x) = \Sigma_i u_i(x_i)$ and xPy iff $u(x) > u(y) + 1$. Then P is obviously consistent, monotone and continuous.

Note that this P is representable, by construction. □

Lemma 5.3. Let P be a semiorder which is monotone, and representable by u. Then there exists $f_u: \mathscr{R}^n \to \mathscr{R}$ such that $u = f_u(u_1, \ldots, u_n)$, i.e. $u_i(x_i) = u_i(y_i)$, $i = 1, \ldots, n$, imply $u(x) = u(y)$. Moreover, f_u is unique.

Proof. Suppose that $u(x) > u(y)$, i.e. xQy. Since u is onto $\mathscr{R}$, there exists z in X such that

$$u(y) \leq u(z) + 1 \text{ and } u(x) > u(z) + 1;$$

Note that $y_i E_i x_i$ and, in particular, $y_i Q_i^0 x_i$, $i = 1, \ldots, n$. Hence, by monotonicity, yPz, which contradicts the condition $u(y) \leq u(z) + 1$. □

In Gilboa and Lapson (1990) we discuss relations between properties of P, and, in particular, show independence of our monotonicity and consistency. We are now in a position to prove Theorem A.

Proof of the Theorem A. Let us first show that (ii) implies (i). Assume, then, that $u = f_u(u_1, \ldots, u_n)$ with f_u as in (ii). Monotonicity of P follows from monotonicity of f_u. P has to be consistent because of (2) and strong monotonicity of f_u. Finally, let us show that the continuity of f_u implies that of P. Assume that a sequence $\{x_{ik}\}$ converges to x_i as $k \to \infty$ in the P_i-topology on X_i. Since range $(u_i) = \mathscr{R}$, this implies that $u_i(x_{ik}) \to u_i(x_i)$ as $k \to \infty$. By continuity of f_u, $u(x_k)$ converges to $u(x)$ which implies that x_k converges to x in the P-topology.

We now wish to show that (i) implies (ii). By Lemma 5.3, we know that monotonicity of P implies the existence of a unique $f_u: \mathscr{R}^n \to \mathscr{R}$ such that $u = f_u(u_1, \ldots, u_n)$. We will now prove it satisfies all requirements.

Strict monotonicity. Let $a_i, b_i \in \mathscr{R}$ satisfy $a_i > b_i$. We will show that for every $c_{-i} \in \mathscr{R}^{n-1} f_u(c_{-i}, a_i) > f_u(c_{-i}, b_i)$. Since range $(u_i) = \mathscr{R}$, we can find $x, y, z \in X$ such that $u_i(x_i) = a_i$, $u_i(y_i) = b_i$ and $b_i - 1 < u_i(z_i) < a_i - 1$. Similarly, let $w_j \in X_j$, $(j \neq i)$ satisfy $u_j(w_j) = c_j$. Note that xP_iz but $\neg(yP_iz)$. By consistency, $(w_{-i}, x_i)P(w_{-i}, z_i)$ and $\neg((w_{-i}, y_i)P(w_{-i}, z_i))$. Thus, by monotonicity, $(w_{-i}, x_i)Q\,(w_{-i}, y_i)$ and $f_u(c_{-i}, a_i) > f_u(c_{-i}, b_i)$.

Continuity. Assume the contrary, i.e., that f_u is not continuous at some point $a \in \mathscr{R}^n$. Then there is an $\varepsilon > 0$ and a sequence $\{a_k\}$ converging to a as $k \to \infty$ such that $f_u(a_k) > f_u(a) + \varepsilon$ for all k or $f_u(a_k) < f_u(a) - \varepsilon$ for all k. Let us assume the former, i.e., $f_u(a_k) > f_u(a) + \varepsilon$. Find $x_k \in X$ such that $(u_1(x_{1k}), \ldots, u_n(x_{nk})) = a_k$, an $x \in X$ for which $(u_1(x_1), \ldots, u_n(x_n)) = a$ and $z \in X$ such that $f_u(a) + \varepsilon + 1 > u(z) > f_u(a) + 1$. Since $a_k \to a$ as $k \to \infty$ for each i, $u_i(x_{ik}) \to u_i(x_i)$, which implies that $x_{ik} \to x_i$ as $k \to \infty$ in the P_i-topology on X_i. However, for all k $\neg(zPa_k)$ while zPa, so that x_k does *not* converge to x in the P-topology. A contradiction to the continuity of P.

Condition (2). Let there be given a vector $a \in \mathscr{R}^n$ and an index $1 \leq i \leq n$ and consider $f_u(a_{-i}, a_i + 1)$. By consistency, $f_u(a_{-i}, a_i + 1 + \varepsilon) > f_u(a) + 1$ for all $\varepsilon > 0$, and $f_u(a_{-i}, a_i + 1 - \varepsilon) \leq f_u(a) + 1$. (Although this is not necessary for the proof, it is worth noting that strict monotonicity of f_u also implies that the latter inequality is strict, namely that $f_u(a_{-i}, a_i + 1 - \varepsilon) < f_u(a) + 1$ for all $\varepsilon > 0$.) The continuity of f_u implies $f_u(a_{-i}, a_i + 1) = f_u(a) + 1$.

Finally, notice that $u(x) = \Sigma_i u_i(x_i)$ obviously satisfies (ii); hence its associated semiorder satisfies (i). □

Our next objective is to see to what extent semiorders on product spaces which are consistent, monotone and continuous with respect to given ones on the original spaces are unique. First we want to examine the possibility of reverse preference: could it happen that two distinct semiorders P_a and P_b satisfying the conditions of Theorem A rank two alternatives in opposite direction, namely xP_ay and yP_bx? We first consider the case $n = 2$:

Observation 5.4. Let P be a semiorder on X which is consistent, monotone and continuous with respect to P_1 and P_2. Let $x, y \in X$ be such that $u_1(x_1) > u_1(y_1)$ and $u_2(x_2) < u_2(y_2)$, but $\neg(x_1P_1y_1)$ and $\neg(y_2P_2x_2)$. Then $\neg(xPy)$.

Proof. By Theorem A, P may be represented by $u = f_u(u_1, u_2)$. Suppose xPy. Then, by monotonicity, zPy for z in X with $u_1(z_1) = u_1(x_1)$ and $u_2(z_2) = u_2(y_2)$. But, by consistency, $\neg(zPy)$, a contradiction. □

In Gilboa and Lapson (1990) we show that in the case of $n = 3$, preference reversal is also impossible. However, it is possible for large enough n:

Observation 5.5. For any $n > 3$ there are semiorders $P_1, \ldots, P_n$ on $X_1, \ldots, X_n$, P_a, P_b on X and $x, y \in X$ such that

(i) P_a, P_b are both consistent, monotone and continuous with respect to $P_1, \ldots, P_n$;
(ii) xP_ay and yP_bx.

Proof. We provide an example for the case $n = 4$. It can be easily extended to any $n > 4$ (for instance, by setting $x_i = y_i$, for $i > 4$). Set $X_i = \mathscr{R}$, $u_i(x_i) = x_i$, $i = 1, \ldots, 4$. Define $u_a(x) = \Sigma u_i(x_i)$. As usual suppose that all semiorders are induced by corresponding utility functions.

Choose another representation of $P_1, \ldots, P_4$. Let

$$v_i(x_i) = \{(x_i - k)/100 + k, \quad \text{if } x_i \in [k, k + 0.1];$$
$$19.96(x_i - k) + k - 1.995, \quad \text{if } x_i \in]0.1 + k, 0.15 + k[;$$
$$(x_i - k)/850 + k + 849/850, \quad \text{if } x_i \in [0.15 + k, k + 1[\},$$

where k is an integer, $i = 1, \ldots, 4$.

Define $u_b(x) = \Sigma v_i(x_i)$.

By Theorem A, both P_a and P_b are consistent, monotone and continuous with respect to $P_1, \ldots, P_4$. Let $x = (0.1, 0.1, 0.95, 0.95)$ and $y = (0.2, 0.2, 0.3, 0.3)$. Then $u_a(x) = 2.1$, $u_a(y) = 1$. Hence, xP_ay. But $u_b(x) < 0.002 + 2 = 2.002$ and $u_b(y) > 4 \times 0.999 = 3.996$. Hence, yP_bx. □

Hence, we see that "preference reversal" in the sense of xP_ay but yP_bx is possible. However, $x(P_a)^ny$ and $y(P_b)^nx$ is impossible. In fact, our Theorem B shows that for any such P_a and P_b, $(P_a)^n \subseteq P_b$ and $(P_b)^n \subseteq P_a$. Let us turn to its proof.

Proof of Theorem B. Suppose $y(P_a)^nx$. Let x' be an alternative in X such that $u_i(x'_i) = u_i(x_i) + 1$ for $1 \leq i \leq n$, so that $f_u(u_1(x'_1), \ldots, u_n(x'_n)) = f_u(u_1(x_1), \ldots, u_n(x_n)) + n$ for every f_u that corresponds to a representable semiorder P on X which is consistent, continuous and monotone w.r.t. $P_1, \ldots, P_n$.

By Theorem A, yE_az and yE_bz for any $z \in X$ with $u_i(z_i) = u_i(y_i) + m_i$, where $m_i \in Z$ and $\Sigma m_i = 0$. For each $z \in X$ define $d^z = (d^z_1, \ldots, d^z_n) \in \mathscr{R}^n$ by $d^z_i = u_i(z) - u_i(x')$. One can find a z with the following properties:

(i) yE_az and yE_bz
(ii) (a) $d^z_i > 0$ for all $1 \leq i \leq n$
or (b) $d^z_i \leq 0$ for all $1 \leq i \leq n$ (but not (a))
or (c) $|d^z_i| < 1$ for all $1 \leq i \leq n$ (but not (a) nor (b))

Such a z would be, for instance, one minimizing $\Sigma |d^z_i|$ over the set $\{z | u_i(z) - u_i(x) = m_i$, where $m_i \in Z$ and $\Sigma m_i = 0\}$.

In case (ii) (a) we have $u_i(z) \geq u_i(x')$ whence $z_i Q_i^0 x_i'$ and, by monotonicity, $zQ_b^0 x'(P_b)^n x$, which implies $z(P_b)^n x$ and $y(P_b)^n x$.

In case (ii) (b) $x_i' Q_i^0 z_i$ whence $x' Q_a z$ and $x' Q_a y$. However, $f_u(u_1(y_1), \ldots, u_n(y_n)) \geq f_u(u_1(x_1), \ldots, u_n(x_n)) + n$ whence we also get $yE_a x'$ and $zE_a x'$. But this is possible only if $d_i^z = 0$ for all i which boils down to (ii) (a).

Finally, consider case (ii) (c). Since $d_i^z > -1$ we know that $u_i(z) > u_i(x)$ for all i. However, we also know that for some i $d_i^z > 0$ which means that $u_i(z_i) > u_i(x_i) + 1$. Monotonicity and consistency of P_b mean $zP_b x$, whence $yP_b x$ also holds. □

Corollary 5.6. Let $\{P_i\}_{i \in N}$ on $\{X_i\}_{i \in N}$ be representable semiorders. Suppose that P_a and P_b are two representable semiorders on X which are both consistent, monotone and continuous with respect to $P_1, \ldots, P_n$. Then $I_a \subseteq (I_b)^n$ and $I_b \subseteq (I_a)^n$.

The proof follows from Theorem B and the fact that for representable semiorders $I(P^n)$ induced by P^n coincides with I^n, where I is induced by P. (Recall, that "representability" implies that the range of the utility is all of $\mathcal{R}$.)

Let us turn to the proof of Theorem C.

Lemma 5.7. If two semiorders P and P' satisfy A2, then $Q = Q'$.

Proof. Let k and m satisfy $(P)^k = (P')^m$. $(P)^k$ is a semiorder on X and so is $(P')^m$. Since they are identical, the weak orders $(Q)^k$ and $(Q')^m$ are also identical. However, $(Q)^k = Q$ and $(Q')^m = Q'$. □

For two given representable semiorders P and P' on the same space X define:

$$A(P, P') = \{k/m : k, m \in N,\ P^k \subseteq (P')^m\}.$$

Lemma 5.8. Suppose P and P' are representable semiorders on a space X satisfying A1 and A2. Then $A(P, P')$ is homogeneous, i.e., for every $t \in N$ $P^k \subseteq (P')^m$ iff $P^{tk} \subseteq (P')^{tm}$.

Proof. Throughout the proof let u and u' represent P and P' respectively.

"Only if" part.

$P^k \subseteq (P')^m$ means that for any x, y in X, $u(x) - u(y) > k$ implies $u'(x) - u'(y) > m$. If $x^1 P^{tk} x^{t+1}$ then there exists a sequence $(x^1, \ldots, x^t)$ in X such that $x^i P^k x^{i+1}$ for all $i = 1, \ldots, t$. Then $u(x^i) - u(x^{i+1}) > k$ for all $i = 1, \ldots, t$. This, in turn, implies $u'(x^i) - u'(x^{i+1}) > m$ for all $i = 1, \ldots, t$, or $x^1 (P')^{tm} x^{t+1}$.

"If" part.

Assume, then, that $P^{tk} \subseteq (P')^{tm}$ for some $t > 1$. Since A2 holds there are two possible cases: 1) $P^k \subseteq (P')^m$, in which the proof is complete, or 2) $(P')^m \subset P^k$. In this case, by the only if part, $(P')^{tm} \subseteq P^{tk}$. But, by assumption, $P^{tk} \subseteq (P')^{tm}$. Thus, $P^{tk} = (P')^{tm}$.

To show that $P^k \subseteq (P')^m$, let there be given $x, y \in X$ with $xP^k y$; we will show that $x(P')^m y$ has to hold. Suppose not. Then $u(x) - u(y) > k$ but $u'(x) - u'(y) \leq m$. Choose a sequence $y^0, \ldots, y^t \in X$ with $y^0 = x$ and $u(y^i) - u(y^{i+1}) = k$ for $0 \leq i \leq t - 1$. This is possible since range $(u) = \mathcal{R}$. Note that $u'(x) - u'(y^1) < m$ since $y^1 Q y$ and this is equivalent to $y^1 Q' y$.

By our construction, $\neg(y^iP^ky^{i+1})$ for $0 \le i \le t-1$ which implies, since $(P')^m \subseteq P^k$, that $\neg(y^i(P')^my^{i+1})$. However, for every z satisfying y_tQz we get – again, using the fact that range $(u) = \mathscr{R}$, – $xP^{tk}z$. The latter means that $x(P')^{tm}z$.

Considering the u' scale, we obtain $u'(x) - u'(z) > tm$ for every z satisfying y^tQz (equivalently, $y^tQ'z$). Hence, $u'(x) - u'(y^t) \ge tm$. On the other hand, $\neg(y^i(P')^my^{i+1})$ for $0 \le i \le t-1$ implies $u'(y^i) - u'(y^{i+1}) \le m$ whence $u'(x) - u'(y^t) \le tm$. Combining the inequalities one obtains $u'(x) - u'(y^1) = m$ in contradiction to the choice of y^i. □

We now proceed to our third main result, Theorem C.

Proof of Theorem C. *(i)⇒(ii)*

Let us begin with A2. Consider P_i, P_j and $k, m \in N$. By (i), $x(P_i)^ky$ iff $u(x) - u(y) > k\delta_i$, whence $(P_i)^k \subseteq (P_j)^m$ iff $k\delta_i \le m\delta_j$. Hence, $(P_i)^k \subseteq (P_j)^m$ or $(P_j)^m \subseteq (P_i)^k$. Since $\{\delta_i\}$ are rational equality would hold for some $k, m \in N$.

As for A1, note that $x(P_iP_j)y$ iff $u(x) - u(y) > \delta_i + \delta_j$, which means that $P_iP_j = P_jP_i$.

(ii)⇒(i)

Let us first introduce some additional definitions. For a semiorder P on X, let P^* be the binary relation defined as follows: xP^*y iff xIy and for every z satisfying zQx we have zPy. Intuitively, xP^*y means that x is the "*supremum*" of $\{w \mid wIy\}$. By the usual composition of binary relations $(P^*)^k$ is well-defined for $k \ge 1$. Let us also define $(P^*)^0$ to be E (which corresponds to equal u-values) and $(P^*)^{-k}$ for $k \ge 1$ as the inverse of $(P^*)^k$. Similarly, we will refer to the expressions of the type $(P^*_{i_1})^{k_1} \ldots (P^*_{i_s})^{k_s}$, where $i_r \in \{1, \ldots, n\}$ and $k_r \in Z$ for $1 \le r \le s$.

The proof will be simpler to carry out by induction on n. Let us begin with $n = 2$.

Choose any point x_0 in X and set $u(x_0) = 0$. By A2, there are $m, t \in N$ such that $(P_1)^m = (P_2)^t$. Assume without loss of generality that g.c.d. $(m, t) = 1$, where g.c.d. stands for the greatest common divider. This assumption can be made thanks to Lemma 5.8. Define $M = t*m$. We will construct a function u such that

$$u(x) - u(y) > \delta_1 \equiv t \quad \text{iff } xP_1y \quad \text{and}$$
$$u(x) - u(y) > \delta_2 \equiv m \quad \text{iff } xP_2y \tag{3}$$

For every integer k let us define $V(k) = \{y \in X \mid$ there exist sequences $k_1, \ldots, k_s$ and $i_1, \ldots, i_s$ such that $y(P^*_{i_1})^{k_1} \ldots (P^*_{i_s})^{k_s}x_0$ and $\Sigma k_r\delta_{i_r} = k\}$. Intuitively, $V(k)$ is the set of all y's for which we have to assign the value $u(y) = k$. Note that $V(k) \ne \emptyset$ for every $k \in Z$.

Claim 1. For every k and every $y, z \in V(k)$ it is true that yEz.

Proof. For $k \in Z$ there are unique a_k, b_k, c_k such that $k = a_kM + b_kt + c_km$ with $a_k \in Z$, $0 \le b_k \le m$ and $0 \le c_k \le t$. Note that A1 and A2 imply that $(P^*_1)^m = (P^*_2)^t$ and that $P^*_1P^*_2 = P^*_2P^*_1$. Hence, every $y \in V(k)$ satisfies $y(P^*_1)^{ma_k + b_k}(P^*_2)^{c_k}x_0$, which implies the desired conclusion. □

Claim 2. Suppose $y \in V(k)$ and $z \in V(g)$ with $k > g$. Then yQz.

Proof. Since u_i represents P_i $(i = 1, 2)$, for every $w_1, w_2, t_1, t_2 \in X$, if $w_1P^*_iw_2$ and $t_1P^*_it_2$ $(i = 1, 2)$ then w_1Qt_1 iff w_2Qt_2. Using this argument inductively, for every

$w_1, w_2, t_1, t_2 \in X$, and every $j, h \in Z$, if $w_1(P_1^*)^j (P_2^*)^h w_2$ and $t_1(P_1^*)^j (P_2^*)^h t_2$ then $w_1 Q t_1$ iff $w_2 Q t_2$.

Consider $g = k - 1$. There are j and h such that $jt + hm = -1$. For $y \in V(k)$ and $z \in V(g)$ choose $w \in V(g-1)$. Then $y(P_1^*)^j (P_2^*)^h z$ and $z(P_1^*)^j (P_2^*)^h w$. Hence, yQz iff zQw. It turns out that one of the following is true:

(i) for every k, g, $y \in V(k)$ and $z \in V(g)$ with $k > g$ implies yQz.
(ii) for every k, g, $y \in V(k)$ and $z \in V(g)$ with $k > g$ implies zQy.
(iii) for every k, g, $y \in V(k)$ and $z \in V(g)$ with $k > g$ implies yEz.

One only needs to know that for $k = t$ and $g = 0$ $y \in V(k)$ and $z \in V(0)$ satisfy yQz to conclude that (i) is the case. □

At this point one can define u on $\bigcup_{k \in z} V(k)$ by $u(y) = k$ for $y \in V(k)$. It is obvious that u satisfies (3) for $x, y \in \bigcup_{k \in z} V(k)$.

Next, choose $x_1 \in V(1)$. Denote $I = \{x \mid x_1 Q x Q x_0\}$. For every x in I define $u(x) = u_1(x)/u_1(x_1)$. For every $k \in Z$ define a set $V(x, k)$ as $V(k)$ was defined for $x = x_0$. Note that for $y \in V(x, k)$ and $z \in V(k)$, $w \in V(k+1)$ we have $wQyQz$. Furthermore, for every $y \in X$ there are $x \in I$ and $k \in Z$ such that $y \in V(x, k)$. Hence, we define $u(y) = k + u(x)$.

It is easy to see that for every $x, y \in X$ and $i = 1, 2$ $xP_i^* y$ iff $u(x) - u(y) = \delta_i$.

We now turn to the induction step. Suppose $n > 2$. We already know that for $P_1, \ldots, P_{n-1}$ there is a function u and positive rational numbers $\delta_1, \ldots, \delta_{n-1}$ such that $xP_i y$ iff $u(x) - u(y) > \delta_i$ and xQy iff $u(x) > u(y)$. Without loss of generality assume that $\delta_i \in N$. Define P' by $xP'y$ iff $u(x) - u(y) > 1$, so that $P_i = (P')^{\delta_i}$ for $1 \le i \le n-1$. Let u', δ' and δ'_n represent P' and P_n, namely,

$$u'(x) - u'(y) > \delta' \quad \text{iff } xP'y;$$
$$u'(x) - u'(y) > \delta'_n \quad \text{iff } xP_n y;$$
$$\text{and } u'(x) - u'(y) > 0 \quad \text{iff } xQy,$$

for every $x, y \in X$ (The existence of those is guaranteed by the proof for the case $n = 2$.). Furthermore, δ' and δ'_n may be assumed to be integer w.l.o.g. Hence, u' also satisfies $u'(x) - u'(y) > \delta'_i \equiv \delta_i * \delta'$ iff $xP_i y$ for $1 \le i \le n$. This completes the proof of the theorem. □

Finally, we note that axioms A1 and A2 are independent.

Observation 5.9. A2 does not imply A1.

Proof. Consider the following example: $n = 2$, $X = \mathscr{R}$, $u_1(x) = x$,

$$u_2(x) = \begin{cases} 2k + 5/8(x - 3k), & \text{if } 3k \le x < 3k + 2, \quad k \in Z; \\ 2k + 5/4 + 3/4(x - 3k - 2), & \text{if } 3k + 2 \le x < 3k + 3, \quad k \in Z. \end{cases}$$

Define $xP_1 y \Leftrightarrow u_1(x) - u_1(y) > 1$ and $xP_2 y \Leftrightarrow u_2(x) - u_2(y) > 1$.
P_1 and P_2 satisfy A2: $(P_1)^3 = (P_2)^2$ and for every $k, l \in N$ $k \ge (3/2)l$ implies $(P_1)^k \subseteq (P_2)^l$. However, to see that A1 fails to hold take $x = 1$, $z = 3.45$. $zP_2 P_1 x$ but $\neg(zP_1 P_2 x)$. □

Observation 5.10. A1 does not imply A2.

Proof. Again consider ar ^ex^ample with $n = 2$, $X = \mathscr{R}$, $u_1(x) = x$. For $0 \leq x \leq 2$ define

$$u_2(x) = \begin{cases} (1/2)x, & \text{if } 0 \leq x < 0.1; \\ 0.05 + 2(x - 0.1), & \text{if } 0.1 \leq x < 0.2; \\ 0.25 + (x - 0.2)/2, & \text{if } 0.2 \leq x < 0.3; \\ x, & \text{if } 0.3 \leq x \leq 2.\} \end{cases}$$

Extend u_2 to $\mathscr{R}$ in the unique way that will satisfy

$$u_2(x+1) + 1 = u_2(u_2^{-1}[u_2(x) + 1] + 1)$$

(it is easy to see that there exists a unique continuous and strongly monotone u_2 which satisfies this condition).

Finally, define P_1 and P_2 by u_1 and u_2 respectively with a justnoticeable difference of 1.

By definition, $P_1P_2 = P_2P_1$. However, A2 does not hold: for $x = 0.1$ and $y = 1.12$ we have yP_1x but $\neg(yP_2x)$ while for $z = 0.2$ and $w = 0.76$ zP_2w holds while zP_1w does not. □

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