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A Game-Theoretic Approach to the Binary Stochastic Choice Problem

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We provide an equivalence theorem for the binary stochastic choice problem, which may be thought of as an implicit characterization of binary choice probabilities which are consistent with a probability over linear orderings. In some cases this implicit characterization is very useful in derivation of explicit necessary conditions. In particular, we present a new set of conditions which generalizes both Cohen and Falmagne's and Fishburn's conditions.

1. INTRODUCTION

The binary stochastic choice problem is the following: given a set N of alternatives and numbers, $\{p_{ij}\}_{i, j \in N; i \neq j}$, interpreted as "the probability that i will be preferred to j," when is there a probability distribution Pr on the linear orderings of N, $\{R^m\}_{m=1}^{|N|!}$, such that

$$p_{ij} = \sum_{\{m \mid iR^m j\}} \Pr(R^m)?$$

(In this case, $\{p_{ij}\}$ will be called *consistent.*)

We will not expatiate here on the motivation, background, or history of the problem. We refer the reader to Block and Marschak (1960), McFadden and Richter (1970), Falmagne (1978), Cohen and Falmagne (1990), Dridi (1980), Souza (1983), McLennan (1984), Barbera (1985), Barbera and Pattanaik (1986),

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Fishburn (1988), Fishburn and Falmagne (1988), Marley (1989), Gilboa (1990), and Koppen (1990).

While it is known that for every $n=|N|$ there are finitely many linear inequalities in $\{p_{ij}\}_{ij}$ which fully characterize the consistent binary choice probabilities, there is no set of explicit conditions which are necessary and sufficient for all n. Several necessary conditions are, however, known, and they are quoted in Section 2.

Following Monderer (1992), which uses a game-theoretic approach to derive Block–Marschak conditions for the (non-binary) stochastic choice problem, this paper uses a similar approach to derive an equivalence theorem which may be considered as an implicit characterization. Although it falls short of an explicit one, i.e., it does not provide explicit formulae for finitely many linear inequalities, it may be used to derive necessary conditions. In Section 3 we state and prove the equivalence theorem, and Section 4 shows how it can be used to derive some known conditions.

These proofs show that in some cases the equivalence theorem is a useful tool in proving necessity of conditions, which is more technical and requires less imagination than direct combinatorial proofs. Indeed, the proofs in Section 4 suggested natural generalizations, and in Section 5 we present a new set of necessary conditions, which unifies and generalizes the Cohen–Falmagne conditions on one hand and the Fishburn conditions on the other. We also present another new set of conditions, which was also developed by Koppen (1990).

However, we also have some bad news. In trying to obtain the diagonal inequality, which is a generalization of the Cohen–Falmagne conditions, we were not very successful at utilizing the equivalence theorem. It seems that in this case the direct combinatorial proof is significantly simpler than the one using the theorem. In Section 6 we discuss the diagonal inequality and provide a new combinatorial proof of it, which may be insightful by itself, as well as the additional computations needed to formally employ the equivalence theorem.

Finally, Section 7 concludes this paper with a remark on the insufficiency of the known conditions.

2. Known Necessary Conditions

2.1. *The Triangle Inequality*

This conditions, which is to be found in Block and Marschak (1960), is a direct implication of transitivity. It says that for every $i, j, k \in N$

$$p_{ij}+p_{jk}+p_{ki}\leqslant 2.$$

2.2. *Cohen and Falmagne's Inequality*

In Cohen and Falmagne (1990), we find the following condition: for every two sequences, $A=(a_1, a_2, ..., a_k)$ and $B=(b_1, b_2, ..., b_k)$, where $\{a_i\}_{i=1}^{k} \cap \{b_i\}_{i=1}^{k}=\varnothing$,

$$\sum_{\{(i,j)\mid 1\leqslant i\neq j\leqslant k\}} p_{a_ib_j}-\sum_{i=1}^{k} p_{a_ib_i}\leqslant k(k-2)+1.$$

(Here and in the sequel, a "sequence" refers to a sequence of distinct elements of N.)

2.3. *Fishburn's Condition*

Fishburn (1988) provides the following necessary conditions: for every two sequences, $A=(a_1, a_2, ..., a_k)$ and $B=(b_1, b_2, ..., b_k)$, with $\{a_i\}_{i=1}^k \cap \{b_i\}_{i=1}^k = \varnothing$ and $k=2l-1$,

$$\sum_{i=1}^{k} p_{a_i b_i} + \sum_{i=1}^{k} p_{a_i b_{i+1}} - \sum_{i=1}^{k} p_{a_i b_{i+l}} \leqslant 3l-2,$$

where the addition operation on indices is done mod k.

It should be mentioned at this point that McLennan's paper includes two conditions which are very similar to Cohen and Falmagne's and to Fishburn's conditions, respectively.

2.4. *The Diagonal Inequality*

Gilboa (1990) proved the following to be a necessary condition: for any two sequences, $A=(a_1, a_2, ..., a_k)$ and $B=(b_1, b_2, ..., b_k)$ (not necessarily disjoint), and every $1 \leqslant r \leqslant k-1$,

$$\sum_{\{(i,j) \mid 1 \leqslant i \neq j \leqslant k\}} p_{a_i b_j} - r \sum_{i=1}^{k} p_{a_i b_i} \leqslant k(k-1) - rk + r(r+1)/2.$$

This condition is identical to Cohen and Falmagne's in the case $r=1$ and disjoint sequences. With $A=(i, j)$ and $B=(j, k)$ it is equivalent to the triangle inequality (assuming, w.l.o.g. (without loss of generality), that $p_{ij}+p_{ji}=1$ and setting $p_{ii}=0$ for all $i \in N$).

3. The Equivalence Theorem

We first introduce some game-theoretic definitions. Given a finite and nonempty set (of *players*) N, we define a *game* v on N to be a set function $v: 2^N \to \mathbb{R}$ with $v(\varnothing)=0$. (Subsets of N are interpreted as *coalitions.*) For any $\varnothing \neq T \subseteq N$, we define v_T to be the *unanimity game* on T by

$$v_T(S) = \begin{cases} 1 & S \supseteq T \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that $\{v_T\}_{T \subseteq N, T \neq \varnothing}$ is a basis for the linear space of games on N (endowed with the natural linear operations).

Given a game v, and a player i, we define i's *maximal marginal contribution* in v to be

$$v_i^* = \max\{v(S \cup \{i\}) - v(S) \mid S \subseteq N, i \notin S\}.$$

A convenient abuse of notation is to identify a player i with the singleton $\{i\}$, and we will enthusiastically do so whenever possible.

We can now formulate:

THE EQUIVALENCE THEOREM. *Given a finite and nonempty set* N, *and numbers* $\{p_{ij}\}_{i,j\in N;\, i\neq j}$, *the following are equivalent*:

(i) $\{p_{ij}\}$ *are consistent*;

(ii) *For every* $\{\alpha_{ij}\}_{i,j\in N;\, i\neq j}$ *and every game* u,

$$\sum_{i,j\in N;\, i\neq j} \alpha_{ij} p_{ij} \leqslant \sum_{i\in N} (v^i - u)^*_i + u(N),$$

where

$$v^i = \sum_{\{j \mid j\neq i\}} \alpha_{ij} v_{\{i,j\}}.$$

Before proving this theorem, we would like to compare this result to the following:

THE CHARACTERIZATION THEOREM. *Given a finite and nonempty set* N, *and numbers* $\{p_{ij}\}_{i,j\in N;\, i\neq j}$, *the following are equivalent*:

(i) $\{p_{ij}\}$ *are consistent*;

(ii) *for every* $\{\alpha_{ij}\}_{i,j\in N,\, i\neq j}$,

$$\sum_{i,j} \alpha_{ij} p_{ij} \leqslant \max_R \sum_{iRj} \alpha_{ij}$$

(*where the* max *is taken over all linear orderings* R *of* N).

The characterization theorem basically says that a point (p_{ij}) is in the convex hull of other points (the 0–1 probabilities which correspond to the linear orderings) if and only if it satisfies every linear inequality satisfied by these points. This fact, which is well known by now, dates back to Weyl (1935) at the latest. In the context of the binary choice problem, see McFadden and Richter (1970) and Fishburn (1988).

While both the characterization and the equivalence theorems provide implicit characterizations of consistent probabilities, they tend to be quite different in practical use: suppose one is given coefficients $\{\alpha_{ij}\}$ and one would like to compute

$$\beta \equiv \max_R \sum_{iRj} \alpha_{ij},$$

i.e., the tightest necessary condition involving these $\{\alpha_{ij}\}$. Every game u may be plugged into the equivalence theorem to generate a necessary condition, which does not have to be the tightest one. Thus, it provides an *upper bound* on β. On the other hand, every linear ordering R can be used to compute a *lower bound* on β. (Note, however, that the resulting condition will typically not be a necessary one as it may be too strict.)

Thus, the following procedure suggests itself: Choose a set of coefficients $\{\alpha_{ij}\}$. (This part, alas, requires imagination regardless of the theorem one chooses to work with.) Find games u and orderings R until the upper bound implied by u (via the equivalence theorem) equals the lower bound implied by R (via the characterization theorem).

Sections 4 and 5 provide several examples of this method (we omit the computation of the lower bound for known conditions, which are also known to be tight.) In these examples, the proof that a certain condition is necessary will be greatly simplified by the equivalence theorem, which basically reduces it to the computation of

$$\sum_{i \in N} (v^i - u)_i^* + u(N). \qquad (*)$$

Indeed, a certain amount of originality may be required for the specification of the appropriate u, just as it is required for the choice of the ordering R to be used in the characterization theorem. Yet the proof may be much simpler (and shorter) than a proof which relies on one of the theorems alone: one need not show that *any* other game, u', cannot yield a lower value for (*), nor that *any* other ordering R' cannot provide a higher value for $\sum_{iR'j} \alpha_{ij}$.

It is interesting to note, though, that if one fails to find the appropriate u and R, applications of the equivalence and characterization theorems become equivalent. As will be clear from the proof, the "best" game u is given by

$$u(S) = \max_R \sum_{iRj;\, i,\, j \in S} \alpha_{ij}$$

and it satisfies

$$(v^i - u)_i^* = 0.$$

Hence, one can always compute β by

$$\sum_{i \in N} (v^i - u)_i^* + u(N) = u(N) = \max_R \sum_{iRj} \alpha_{ij}.$$

In Section 6 we provide a new proof of the diagonal inequality, which we find interesting. However, from the point of view of the equivalence theorem, Section 6 is a report of a failure: we were not imaginative enough to find appropriate u and R. Hence, we resorted to a direct computation of $\max_R \sum_{iRj} \alpha_{ij}$. As explained above, this computation may be viewed as an application of either the equivalence theorem *or* the characterization theorem. It is precisely the failure to use both theorems *together* that necessitates a direct combinatorial proof.

We now proceed to prove the equivalence theorem. As a matter of fact, it is a special case of Theorems A and B in Gilboa and Monderer (1991). For the sake of completeness, however, we provide here a proof as well:

(a) (i) $\Rightarrow$ (ii). Suppose $\{\alpha_{ij}\}$ and u are given. It suffices to show that for every given linear ordering R on N the probabilities $\{p_{ij}^R\}$ defined by

$$p_{ij}^R = \begin{cases} 1 & iRj \\ 0 & \text{otherwise} \end{cases}$$

satisfy (ii). W.l.o.g., assume that $N = \{1, 2, ..., n\}$ and that R is the natural ordering. We therefore need to show that

$$\sum_{i>j} \alpha_{ij} \leqslant \sum_i (v^i - u)_i^* + u(N).$$

For each i, let $S^i = \{j \mid j < i\}$. By definition of the * operation,

$$\begin{aligned}(v^i - u)_i^* &\geqslant (v^i - u)(S^i \cup i) - (v^i - u)(S^i) \\ &= [v^i(S^i \cup i) - v^i(S^i)] - [u(S^i \cup i) - u(S^i)].\end{aligned}$$

As $v^i(T) = \sum_{\{j \mid j \neq i;\, j \in T\}} \alpha_{ij}$ if $i \in T$, and $v^i(T) = 0$ if $i \notin T$, we obtain

$$v^i(S^i \cup i) = \sum_{\{j \mid j < i\}} \alpha_{ij}$$

and

$$v^i(S^i) = 0.$$

Hence,

$$\begin{aligned}&\sum_i (v^i - u)_i^* + u(N) \\ &\quad \geqslant \sum_i \sum_{\{j \mid j < i\}} \alpha_{ij} - \sum_i [u(S^i \cup i) - u(S^i)] + u(N) \\ &\quad = \sum_{j<i} \alpha_{ij}.\end{aligned}$$

We now wish to show the converse.

(b) (ii) $\Rightarrow$ (i). In view of the characterization theorem, it suffices to show that for given $\{\alpha_{ij}\}_{i,j}$ there is a game u such that $\sum_{i \in N} (v^i - u)_i^* + u(N) = \beta = \max_R \sum_{iRj} \alpha_{ij}$ (where v^i are defined as in the statement of the theorem). Define a game u by

$$u(S) = \max_R \sum_{iRj;\, i,j \in S} \alpha_{ij}.$$

Condition (ii) implies that

$$\sum_{i \neq j} \alpha_{ij} p_{ij} \leqslant \sum_{i} (v^i - u)_i^* + u(N),$$

where $v^i = \sum_{\{j \mid j \neq i\}} \alpha_{ij} v_{\{i, j\}}$ and $u(N) = \beta$. Hence, all we need to show is that

$$(v^i - u)_i^* = 0$$

for all $i \in N$. However, $v^i(i) = u(i) = 0$, so that $(v^i - u)_i^* \geqslant 0$ is obvious. To show the converse inequality we have to convince ourselves (and the reader) that for every $S \subseteq N$ and every $i \notin S$

$$v^i(S \cup i) - v^i(S) \leqslant u(S \cup i) - u(S).$$

Note that

$$v^i(S \cup i) - v^i(S) = \sum_{j \in S} \alpha_{ij}.$$

Let R_S be a linear ordering such that

$$u(S) = \sum_{\{k,\, j \in S;\, k R_S j\}} \alpha_{kj}.$$

Let R'_S be a linear ordering which agrees with R_S on S and satisfies $iR'_S j$ for $j \in S$. Then

$$u(S \cup i) \geqslant \sum_{\{i,\, j \in S \cup i;\, iR'_S j\}} \alpha_{ij} = u(S) + \sum_{j \in S} \alpha_{ij}$$

which completes the proof.

4. Derivation of Known Conditions

In order to get used to the game-theoretic machinery and illustrate its applicability, we devote this section to the derivation of some known conditions.

4.1. *The Trivial Conditions*

It is usually assumed that the binary probabilities $\{p_{ij}\}$ satisfy $p_{ij} + p_{ji} = 1$ and $p_{ij} \geqslant 0$. Since this was not explicitly assumed in the equivalence theorem, we conclude that these linear conditions are also derived from it. Indeed, let us first choose for some $i, j \in N$ $(i \neq j)$ $\alpha_{ij} = 1$ and $\alpha_{kl} = 0$ for $(k, l) \neq (i, j)$. Letting $u = 0$ we obtain

$$v^i = v_{\{i, j\}} \qquad v^k = 0 \qquad \forall k \neq i$$

and

$$p_{ij} = \sum_{i \neq j} \alpha_{ij} p_{ij} \leqslant \sum_{i} (v^i)_i^* = 1.$$

Next let us take $\alpha_{ij} = \alpha_{ji} = 1$ and $\alpha_{kl} = 0$ for $\{k, l\} \neq \{i, j\}$, with $u = v_{\{i, j\}}$. Then

$$v^i = v^j = v_{\{i, j\}} \qquad v^k = 0 \qquad \forall k \neq i, j$$

and

$$(v^k - u)_k^* = 0 \qquad \forall k \in N,$$

whence one obtains

$$p_{ij} + p_{ji} \leqslant 1.$$

Similarly, $\alpha_{ij} = \alpha_{ji} = -1$ with $u = -v_{\{i, j\}}$ yields $p_{ij} + p_{ji} \geqslant 1$ which implies $p_{ij} + p_{ji} = 1$ and also $p_{ij} \geqslant 0$.

4.2. *The Triangle Inequality*

For given i, j, $k \in N$ (assumed distinct) we have to show that

$$p_{ij} + p_{jk} + p_{ki} \leqslant 2.$$

This formulation naturally suggests

$$\alpha_{ij} = \alpha_{jk} = \alpha_{ki} = 1.$$

(Here and in the sequel, coefficients which are not specifically mentioned should be taken to be zero.)

These coefficients, in turn, define

$$v^i = v_{\{i, j\}} \qquad v^j = v_{\{j, k\}} \qquad v^k = v_{\{i, k\}}.$$

It only remains to choose

$$u = v_{\{i, j\}} + v_{\{j, k\}} + v_{\{i, k\}} - v_{\{i, j, k\}}.$$

It is easy to verify that

$$(v^l - u)_l^* = 0 \qquad \forall l \in N$$

and

$$u(N) = 2,$$

whence the triangle inequality follows.

4.3. *Cohen and Falmagne's Condition*

Let there be given two disjoint sequences $A=(a_1, ..., a_k)$ and $B=(b_1, ..., b_k)$. We wish to prove that

$$\sum_{\{(i,j)\mid 1\leqslant i\neq j\leqslant k\}} p_{a_ib_j}-\sum_{i=1}^{k} p_{a_ib_i}\leqslant k(k-2)+1.$$

Define $\alpha_{a_ib_j}=1$ for $1\leqslant i\neq j\leqslant k$ and $\alpha_{a_ib_i}=-1$ for $1\leqslant i\leqslant k$. Correspondingly,

$$v^{a_i}=\sum_{\{j\mid j\neq i;\, 1\leqslant j\leqslant k\}} v_{\{a_i,b_j\}}-v_{\{a_i,b_i\}}$$

(and $v^{b_i}=0$ for all i).

Let us define u as follows

$$\begin{aligned} u=&\sum_{i=1}^{k} v_{\{a_i,b_1,b_2,...,b_{i-1},b_{i+1},...,b_k\}}\\ &-\sum_{i=1}^{k} v_{\{a_i,b_1,b_2,...,b_{i-1},b_i,b_{i+1},...,b_k\}}+v_{\{b_1,...,b_k\}}.\end{aligned}$$

One may verify that

$$(v^{a_i}-u)^*_{a_i}=k-2 \qquad \text{for} \quad 1\leqslant i\leqslant k$$

and

$$(v^l-u)^*_l=0 \qquad \text{for} \quad l\notin\{a_i\}_{i=1}^{k}.$$

Noting that $u(N)=1$ we obtain the desired result.

4.4. *Fishburn's Condition*

Given two disjoint sequences $A=(a_1, a_2, ..., a_k)$ and $B=(b_1, b_2, ..., b_k)$ with $k=2l-1$ ($l\in N$), we wish to show that

$$\sum_{i=1}^{k} p_{a_ib_i}+\sum_{i=1}^{k} p_{a_ib_{i+1}}-\sum_{i=1}^{k} p_{a_ib_{i+l}}\leqslant 3l-2.$$

Although the equivalence theorem guarantees the existence of a game u which attains the exact bound on the right side, this game is quite complicated to compute. Instead, we will use a simple game which will bring us close enough.

Naturally, we have

$$\alpha_{a_ib_i}=\alpha_{a_ib_{i+1}}=1, \qquad \alpha_{a_ib_{i+l}}=-1, \qquad \forall 1\leqslant i\leqslant k$$

and

$$v^{a_i} = v_{\{a_i, b_i\}} + v_{\{a_i, b_{i+1}\}} - v_{\{a_i, b_{i+l}\}}$$

for $1 \leqslant i \leqslant k$. ($v^j = 0$ for $j \notin \{a_i\}_{i=1}^k$.)

Define

$$\begin{aligned} u = \sum_{i=1}^{k} v_{\{a_i, b_i, b_{i+1}\}} \\ - \sum_{i=1}^{k} v_{\{a_i, b_i, b_{i+1}, b_{i+k}\}} \\ + \tfrac{1}{2} \sum_{i=1}^{k} v_{\{b_i, b_{i+l}\}}. \end{aligned}$$

It is readily seen that $(v^{a_i} - u)^*_{a_i} = 1$ (for all $1 \leqslant i \leqslant k$). The equality $(-u)^*_{b_i} = 0$ (for $1 \leqslant i \leqslant k$) is slightly trickier but still correct. (For more details, see the proof of the generalized condition in Section 5.) Thus one gets

$$\begin{aligned} \sum_{i=1}^{k} p_{a_i b_i} + \sum_{i=1}^{k} p_{a_i b_{i+1}} - \sum_{i=1}^{k} p_{a_i b_{i+l}} \\ \leqslant (2l-1) + u(N) \\ = (2l-1) + (l - \tfrac{1}{2}) = 3l - \tfrac{3}{2}. \end{aligned}$$

Now we have to resort to extraneous argument to complete the proof: since all the $|N|!$ extreme points of consistent $\{p_{ij}\}$, which are integer-valued, satisfy this inequality, they also satisfy it with $(3l-2)$ in the right side. Hence this is also true of every $\{p_{ij}\}$ in their convex hull and this yields Fishburn's inequality.

5. Derivation of New Conditions

The analysis presented above suggests additional necessary conditions. In particular, we present here two sets of new conditions. The first unifies and directly generalizes Cohen and Falmagne's and Fishburn's conditions. The second, which is also to be found in Koppen (1990), is a different generalization of Cohen and Falmagne's condition and, as pointed out in Koppen (1990), can also be reduced to Fishburn's condition.[1]

[1] The coefficients $\{\alpha_{ij}\}$ involved in the second set of conditions appeared in an earlier version of this paper. However, the upper bounds shown here were developed while we were already aware of Koppen's (1990) results. We are also grateful to Mathieu Koppen who observed that the upper bound could probably be improved. The first set of conditions seems to be entirely new.

5.1. *Condition Set* I

Fishburn's condition may be presented as follows: for disjoint sequences $A=(a_1, ..., a_k)$ and $B=(b_1, ..., b_k)$ with $k=2l-1$,

$$\sum_{i=1}^{k} p_{a_i b_{i+l-1}} + \sum_{i=1}^{k} p_{a_i b_{i+l}} - \sum_{i=1}^{k} p_{a_i b_i} \leqslant 3l-2$$

(by shifting the indices of the sequences B by $(-l)$; as usual, all indices are computed mod k).

That is, for every $a_i \in A$, $p_{a_i b_i}$ appears with a negative sign, while for the two b's "across" from a_i, b_{i+l-1} and b_{i+l}, the corresponding p's appear with a positive sign.

We would like to generalize the "two" above and to note that when it becomes $(k-1)$ the expression on the left side is identical to that of Cohen and Falmagne's condition.

To be more specific, we have to deal separately with odd and even k, i.e., the number of elements in A and in B.

Assume, first, that $k=2m+1$. (Here and in the sequel, all variables denote integers unless otherwise specified.) For some $2 \leqslant d \leqslant k$, let q and r be the nonnegative integers satisfying

$$k=qd+r$$

with $0 \leqslant r < d$. Choose s such that $m-q+1 \leqslant s \leqslant m$. (Note that $2 \leqslant d \leqslant k$ implies $1 \leqslant q \leqslant m$. Hence, $1 \leqslant s \leqslant m$ follows.)

Consider the left side:

$$S = \sum_{i=1}^{k} \sum_{j=-s+1}^{s} p_{a_i b_{i+m+j}} - \sum_{i=1}^{k} p_{a_i b_i}.$$

Trying to estimate the right side of the inequality, we first define

$$v^{a_i} = \sum_{j=-s+1}^{s} v_{\{a_i, b_{i+m+j}\}} - v_{\{a_i, b_i\}}$$

and

$$v^{b_i} = 0$$

for $1 \leqslant i \leqslant k$. (Also, $v^x = 0$ for $x \in N \backslash (A \cup B)$.)

For the game u we try to use the same ideas of Subsections 4.3 and 4.4 above: for each a_i we have the unanimity games of two sets—one including all b_j such that $\alpha_{a_i b_j} = +1$, and one including these as well as b_i with a negative sign. This construction guarantees that the maximal marginal contribution of each a_i (to $(v^{a_i}-u)$) is $(2s-1)$, but then each b_i has a maximal contribution of 1 (to $(-u)$.)

So we would like to add to u as few as possible sets of b's that would efficiently reduce their marginal contribution.

Thus, we define

$$u = \sum_{i=1}^{k} v_{\{a_i, b_{i+m-s+1}, \ldots, b_{i+m+s}\}} - \sum_{i=1}^{k} v_{\{a_i, b_{i+m-s+1}, \ldots, b_{i+m+s}, b_i\}} + \frac{1}{d} \sum_{i=1}^{k} v_{\{b_i, b_{i+q}, b_{i+2q}, \ldots, b_{i+(d-1)q}\}}.$$

For $1 \leqslant i \leqslant k$ we indeed obtain $(v^{a_i} - u)^*_{a_i} = 2s - 1$.

As for $(-u)^*_{b_i}$, notice that b_i can only have a positive contribution in the presence of $\{b_{i+m-s+1}, \ldots, b_{i+m+s}\}$. However, since $s \geqslant m - q + 1$, if j satisfies

$$(j-i)(\operatorname{mod} k),\ (i-j)(\operatorname{mod} k) \geqslant q,$$

then $b_j \in \{b_{i+m-s+1}, \ldots, b_{i+m+s}\}$. This implies that there are exactly d indices $v_1, \ldots, v_d$ such that

$$b_i \in \{b_{v_t}, b_{v_t+q}, \ldots, b_{v_t+(d-1)q}\} \subseteq \{b_{i+m-s+1}, \ldots, b_{i+m+s}\}$$

for $t = 1, \ldots, d$. (E.g., $v_t = b_{i-(t-1)q}$). Hence, we get

$$(-u)^*_{b_i} = 0.$$

Finally, $u(N) = k/d$, so we obtain

$$S \leqslant k(2s-1) + k/d.$$

As in the case of Fishburn's condition, we recall that, since $\{\alpha_{ij}\}$ are integers, our β is an integer as well, so we may write

$$\sum_{i=1}^{k} \sum_{j=-s+1}^{s} p_{a_i b_{i+m+j}} - \sum_{i=1}^{k} p_{a_i b_i} \leqslant k(2s-1) + \lfloor k/d \rfloor. \tag{5.1.1}$$

Alternatively, by using $p_{ab} = 1 - p_{ba}$, one obtains

$$\sum_{i=1}^{k} p_{a_i b_i} - \sum_{i=1}^{k} \sum_{j=-s+1}^{s} p_{b_{i+m+j} a_i} \leqslant \lfloor k/d \rfloor = q.$$

To see that this condition is the tighest necessary condition for this set of coefficients, we resort to the characterization theorem and prove by an example that the upper bound q can indeed by obtained. For instance, let us find a linear ordering R such that

$$b_i R a_i \qquad \text{for all} \quad 0 \leqslant i \leqslant q-1$$

and

$$a_i R b_j \qquad \text{for all } 0 \leqslant i \leqslant k-1 \text{ and } 0 \leqslant j \leqslant k-1 \text{ such that}$$
$$(i-j)(\text{mod } k) \geqslant q \text{ and } (j-i)(\text{mod } k) \geqslant q.$$

It is easy to see that these conditions do not contradict transitivity. Hence, such relations R do exist. This concludes the proof for $k = 2m+1$.

Let us now deal with the case $k = 2m$. Let $2 \leqslant d \leqslant k$, $k = qd + r$ with $0 \leqslant r < d$. Choose s such that $m - q \leqslant s \leqslant m-1$ and define

$$S = \sum_{i=1}^{k} \sum_{j=-s}^{s} p_{a_i b_{i+m+j}} - \sum_{i=1}^{k} p_{a_i b_i}.$$

Correspondingly,

$$v^{a_i} = \sum_{j=-s}^{s} v_{\{a_i, b_{i+m+j}\}} - v_{\{a_i, b_i\}}$$

and $v^x = 0$ for $x \notin A$.

Next define

$$\begin{aligned} u = &\sum_{i=1}^{k} v_{\{a_i, b_{i+m-s}, \ldots, b_{i+m+s}\}} \\ &- \sum_{i=1}^{k} v_{\{a_i, b_{i+m-s}, \ldots, b_{i+m+s}, b_i\}} \\ &+ \frac{1}{d} \sum_{i=1}^{k} v_{\{b_i, b_{i+q}, \ldots, b_{i+(d-1)q}\}}. \end{aligned}$$

Similar computations yield

$$\begin{aligned} (v^{a_i} - q)^*_{a_i} &= 2s \\ (-u)^*_{b_i} &= 0 \\ u(N) &= k/d, \end{aligned}$$

whence

$$S \leqslant 2ks + \lfloor k/d \rfloor = 2ks + q \tag{5.1.2}$$

or

$$\sum_{i=1}^{k} p_{a_i b_i} - \sum_{i=1}^{k} \sum_{j=-s}^{s} p_{b_{i+m+j} a_i} \leqslant q.$$

Again, this inequality can easily be seen to be the tighest for this set of coefficients.

In the special case of $k=2l-1=2m+1$, $d=2$ means $q=m$, $r=1$. This allows us to choose $s=1 \geqslant m-q+1$ and the expression on the left side of (5.1.1) reduces to that of Fishburn's condition. Indeed, the right side is

$$k(2s-1)+q=k+q=(2l-1)+(l-1)=3l-2.$$

On the other hand, when $k=2m+1$, taking $d=k$ yields $q=1$, $r=0$, and $s=m$, and the left side of (5.1.1) reduces to that of Cohen and Falmagne's condition. Finally,

$$k(2s-1)+1=k(k-2)+1,$$

which is the corresponding right side.

Similarly, for $k=2m$ we take $d=k$ again, $q=1$, $r=0$, and $s=m-1$. The left side reduces to

$$\sum_{i=1}^{k} \sum_{j \neq i} p_{a_i b_j} - \sum_{i=1}^{k} p_{a_i b_i}$$

and the right side is (in 5.1.2)

$$2ks+q=2k(m-1)+1=k(k-2)+1.$$

Hence, (5.1.1) and (5.1.2) generalize both Fishburn's and Cohen and Falmagne's conditions.

5.2. *Condition Set* II

Another set of necessary conditions can be obtained by considering—again, for disjoint $A=(a_1, ..., a_k)$ and $B=(b_1, ..., b_k)$—expressions of the type

$$S=\sum_{i=1}^{k} \sum_{j=-s}^{-1} p_{a_i b_{i+j}} - \sum_{i=1}^{k} p_{a_i b_i} + \sum_{i=1}^{k} \sum_{j=1}^{s} p_{a_i b_{i+j}}.$$

Let us assume that $k=q(s+1)+r$ with $0 \leqslant r < s+1$, $q \geqslant 1$.

Set

$$v^{a_i} = \sum_{-s \leqslant j \leqslant s;\, j \neq 0} v_{\{a_i, b_{i+s}\}} - v_{\{a_i, b_i\}} \qquad \text{for} \quad i=1, ..., k,$$

$$v^x = 0 \qquad \text{for} \quad x \in N \backslash A$$

and let

$$\begin{aligned} v = & \sum_{i=1}^{k} v_{\{a_i, b_{i-s}, b_{i-s+1}, ..., b_{i-1}, b_{i+1}, b_{i+2}, ..., b_{i+s}\}} \\ & - \sum_{i=1}^{k} v_{\{a_i, b_{i-s}, b_{i-s+1}, ..., b_{i-1}, b_i, b_{i+1}, ..., b_{i+s}\}} \\ & + [1/(s+1)] \sum_{i=1}^{k} v_{\{b_i, b_{i+1}, ..., b_{i+s}\}}. \end{aligned}$$

By considerations similar to those of Subsection 5.1 above, we obtain

$$\begin{aligned}(v^{a_i}-u)^*_{a_i} &= 2s-1\\ (-u)^*_{b_i} &= 0\\ u(N) &= k/(s+1),\end{aligned}$$

whence

$$S \leqslant k(2s-1)+\lfloor k/(s+1)\rfloor = k(2s-1)+q.$$

Or, equivalently,

$$\sum_{i=1}^{k} p_{b_i a_i} - \sum_{i=1}^{k} \sum_{j=-s;\, j\neq 0}^{s} p_{b_{i+j} a_i} \leqslant q. \tag{5.2.1}$$

This condition can easily be seen to be the tighest one for these coefficients, as there are linear orderings R satisfying

$$b_i R a_i \qquad \text{for} \quad i = 1 + t(s+1),\ 0 \leqslant t \leqslant q-1$$

and

$$a_i R b_{i+j} \qquad \text{for all} \quad 1 \leqslant i \leqslant k \text{ and } -s \leqslant j \leqslant s,\ j \neq 0.$$

Note that changing the order of summation in (5.2.1) yields

$$\sum_{i=1}^{k} p_{b_i a_i} - \sum_{i=1}^{k} \sum_{j=-s;\, j\neq 0}^{s} p_{b_i a_{i+j}} \leqslant q = \lfloor k/(s+1)\rfloor,$$

which is identical to Koppen's condition.

Remark. Koppen (1990) also notes that, with a proper permutation of indices, Fishburn's condition is a special case of (5.2.1). It is important to note that neither the same permutation nor any other would reduce (5.1.1) (or (5.1.2)) to (5.2.1) in general. (The case $s=1$ turns out to be a special case of both 5.1.1 and 5.2.1, but setting $s>1$ in either of them yields new conditions.)

6. The Diagonal Inequality

As opposed to the other known conditions, which could be derived from the equivalence theorem relatively easily, and even suggested some generalizations, the diagonal inequality does not seem to be readily obtained by this method. Indeed, the sufficiency proof is constructive enough to specify a game u for any given set of coefficients $\{\alpha_{ij}\}$. However, the computation of the game u used in the proof is more complicated than a direct computation of the right side β: we know that $\beta = u(N)$, so that computing β directly is tantamount to estimating u at one coalition (rather than all coalitions).

However, actual computation of the game u provided by the theorem may be insightful in some cases. In particular, we would like to compute it for the diagonal inequality since the (combinatorial) method of computation may be useful by its own right.

We restrict our attention to the case of disjoint sequences A and B, although this does not exhaust the richness of the diagonal inequality.

Let, then, $A = (a_1, ..., a_k)$ and $B = (b_1, ..., b_k)$ be given with $\{a_i\}_{i=1}^{i} \cap \{b_i\}_{i=1}^{k} = \varnothing$, and let a number $1 \leqslant r \leqslant k-1$ be given. We define

$$\alpha_{a_i b_j} = 1 \qquad 1 \leqslant i \neq j \leqslant k$$

$$\alpha_{a_i b_i} = -r \qquad 1 \leqslant i \leqslant k,$$

which define, for all $1 \leqslant i \leqslant k$,

$$v^{a_i} = \sum_{j=1}^{i-1} v_{\{a_i, b_j\}} - r v_{\{a_i, b_i\}} + \sum_{j=i+1}^{k} v_{\{a_i, b_j\}}$$

(and $v^{b_i} = 0$).

The associated game u is defined by

$$u(S) = \max_R \sum_{\{iRj;\, i, j \in S\}} \alpha_{ij}$$

(where the max is taken over all linear orderings on N or, equivalently, on S).

Given a coalition S, let us assume it contains exactly m pairs (a_i, b_j), l elements of A whose counterpart is in S^c and q elements of B whose counterpart is in S^c. (Thus, $|S \cap \{a_i\}_{i=1}^{k}| = m + l$; $|S \cap \{b_i\}_{i=1}^{k}| = m + q$.)

Let us compute a linear order R which maximizes $\sum_{\{iRj;\, i, j \in S\}} \alpha_{ij}$. Consider an element a_i such that $b_i \in S^c$. Obviously, for every $b_j \in S$, $a_i R b_j$ has to hold for R to be maximal. Similarly, for $b_i \in S$ with $a_i \in S^c$ one has to have $a_j R b_i$ for all $a_j \in S$. The interesting part is, therefore, the m pairs (a_i, b_i). Assume w.l.o.g. that these are $\{(a_i, b_i)\}_{i=1}^{m}$. It is obvious that R may be defined arbitrarily over $\{b_i\}_{i=1}^{m}$. W.l.o.g. assume $b_1 R b_2 R \cdots R b_m$.

The main point in this direct computation method is the following: given the order defined over $\{b_i\}$, each a_i may be separately located in the order R so as to maximize its contribution to the expression $\sum_{i \neq j} p_{a_i b_j} - r \sum_i p_{a_i b_i}$.

Let us now distinguish between two cases: (i) $m \leqslant r$ and (ii) $m > r$. If $m \leqslant r$, it is quite straightforward to verify that the maximal order R has to satisfy $b_i R a_i R b_{i+1}$ (for $1 \leqslant i \leqslant m$). In this case $u(S) = l(m+q) + mq + m(m-1)/2$.

As for case (ii), $b_i R a_i R b_{i+1}$ still has to hold for $1 \leqslant i \leqslant r$. However, for $r+1 \leqslant i \leqslant m$, $a_i R b_1$ is a necessary condition for R's optimality. In this case, one obtains

$$u(S) = l(m+q) + (m-r)(m+q-r-1) \\ + r(m-r+q) + r(r-1)/2.$$

It is also easy to verify that for $S = N$ we obtain $m = k$, $l = q = 0$, and

$$u(N) = k(k-1) - rk + r(r+1)/2.$$

(This proves the diagonal inequality directly and can also be considered an application of the equivalence theorem if we recall that $(v^i - u)_i^* = 0$ for all $i \in N$ and u defined above.)

Worthy of note is the fact that this method of computation of the game u (or of $u(N)$ directly) cannot be applied without some symmetry consideration that would allow assuming an arbitrary order over the sequence B (or part of it). Trying to apply it to Fishburn's condition, for instance, involves combinatorial arguments which are tantamount to a direct proof.

7. A Remark Regarding Sufficiency

None of the explicit conditions mentioned in this paper—the known and the new ones— is sufficient, nor are they sufficient in conjunction. This may be proved by the same example used in Gilboa (1990) to establish the insufficiency of the diagonal inequality. This example involves probabilities $p_{ij} \in \{\frac{1}{3}, \frac{2}{3}\}$, and it is easy to see that for such $\{p_{ij}\}$ all conditions hold. Yet, it was proved that the specific set of $\{p_{ij}\}$ given there is not consistent.

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