Lecture Notes for Introduction to Decision Theory

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Contents

These are notes for a basic class in decision theory. The focus is on decision under risk and under uncertainty, with relatively little on social choice. The notes contain the mathematical material, including all the formal models and proofs that will be presented in class, but they do not contain the discussion of background, interpretation, and applications. The course is designed for 30-40 hours.

1 Preference Relations

A subset of ordered pairs of a set X is called a *binary relation.* Formally, R is a binary relation on X if $R \subset X \times X$.

A binary relation R on X is

– *reflexive* if for every $x \in X$, xRx;

– *complete* if for every $x, y \in X$, xRy or yRx (or possibly both);

– *symmetric* if for every $x, y \in X$, xRy implies yRx;

– *transitive* if for every $x, y, z \in X$, xRy and yRz imply xRz.

Remark 1 *A binary relation that is complete is also reflexive.*

Proof: Given $x \in X$ apply the definition of completeness for the two elements (x, y) being that x. Then xRy or yRx, and in both cases xRx. $\square$

A binary relation R on X is an *equivalence relation* if it is reflexive, symmetric, and transitive. An example of such a relation is equality. Another example is "having the same height" (on a population of people), or "having the same (first) first name". More generally, the equality of any function defines an equivalence relation, and vice versa:

Proposition 2 *A binary relation R on X is an equivalence relation if and only if there exists a set A and a function $f : X \to A$ such that*

$$xRy \qquad \Leftrightarrow \qquad f(x) = f(y) \tag{1}$$

Proof: If such a set and function exist, it is straightforward to verify that R satisfies the three conditions. Conversely, if R is an equivalence relation, one can define

$$A = \{\{y \,|xRy\} \,|x \in X\}$$

and

$$f(x) = \{\, y \mid xRy \,\} \,.$$

To see that (1) holds, assume, first, that xRy. Then $y \in f(x)$ and by symmetry also $x \in f(y)$. Further, transitivity implies that $z \in f(x)$ also satisfies $z \in f(y)$

and vice versa. Thus, $f(x) = f(y)$. Conversely, if $f(x) = f(y)$ we first note that, by reflexivity, $y \in f(y)$, hence $y \in f(x)$ and xRy. $\square$

The set $\{ y \mid xRy \}$ is called the *equivalence class* of x. The set A defined in the proof, namely, the set of all equivalence classes (which obviously defines a partition of X) is called the *quotient set*, denoted X/R.

For a binary relation $\succsim$ on a set of alternatives X, we define the symmetric ($\sim$) and asymmetric ($\succ$) parts as follows. For all $x, y \in X$,

$x \sim y$ if $x \succsim y$ and $y \succsim x$;

$x \succ y$ if $x \succsim y$ and $\neg(y \succsim x)$ (where $\neg$ denotes negation); equivalently, $x \succ y$ if $x \succsim y$ and $\neg(y \sim x)$.

We also use $\precsim$ and $\prec$ for the inverse of $\succsim$ and $\succ$, respectively. That is, $x \precsim y$ is the same thing as $y \succsim x$ and $x \prec y$ is equivalent to $y \succ x$. (Note that the choice of the symbols is supposed to make this natural, but these are new symbols, which denote new relations, so we need to define them.)

Proposition 3 *If $\succsim$ is transitive, then $\succ$ and $\sim$ are transitive.*

Proof: To see that $\sim$ is transitive, assume that $x \sim y$ and $y \sim z$, we have ($x \succsim y$ and $y \succsim x$) as well as ($y \succsim z$ and $z \succsim y$). The first two parts imply (by transitivity of $\succsim$) $x \succsim z$, and the second – $z \succsim x$, so we get $x \sim z$.

To see that $\succ$ is transitive, assume $x \succ y$ and $y \succ z$. That is, ($x \succsim y$ and not $y \succsim x$) as well as ($y \succsim z$ and not $z \succsim y$). The first two parts imply $x \succsim z$ by transitivity of $\succsim$ as above. We need to show that $z \succsim x$ does not hold. Indeed, assume it did. Then we would have $z \succsim x$ and $x \succsim y$, and transitivity (of $\succsim$ again) would imply $z \succsim y$, which is in contradiction to $y \succ z$. Hence $\neg(z \succsim x)$ and $x \succ z$. $\square$

2 Utility Representations

2.1 Representation of a preference order

We say that a function $u : X \to \mathbb{R}$ *represents* a relation $\succsim$ if, for every $x, y \in X$,

$$x \succsim y \qquad \text{iff} \qquad u(x) \geq u(y).$$

That is, we want it to be the case that the relation at-least-as-desirable in terms of observable preferences ($\succsim$) holds between alternatives precisely when their utility values satisfy at-least-as-large in terms of utility numbers ($\geq$).

One can think of other notions of "representation", for instance, requiring that strict inequality between the utility numbers will reflect strict preferences, or that equality would match indifference. It turns out that the first two notions are identical, and that they imply the third:

Theorem 4 *Let $\succsim$ be a complete relation on X and let there be a function $u : X \to \mathbb{R}$. Define*

(i) $x \succsim y$ iff $u(x) \geq u(y)$ $\forall\, x, y \in X$

(ii) $x \succ y$ iff $u(x) > u(y)$ $\forall\, x, y \in X$

(iii) $x \tilde{} y$ iff $u(x) = u(y)$ $\forall\, x, y \in X$

Then (i) and (ii) are equivalent and they imply (iii) (but not vice versa).

Proof: (i) implies (ii): Let there be given $x, y \in X$. Assume first that $x \succ y$. If $u(y) \geq u(x)$, by (i) we have $y \succsim x$, a contradiction to $x \succ y$. Hence, $u(x) > u(y)$. Conversely, assume $u(x) > u(y)$. If $y \succsim x$ we would have (by (i) again) $u(y) \geq u(x)$, which isn't true. Hence $\neg(y \succsim x)$. But completeness implies that $x \succsim y$ or $y \succsim x$ has to hold, and if the latter doesn't hold, the former does. So we have $x \succsim y$ and $\neg(y \succsim x)$, that is, $x \succ y$.

(ii) implies (i): Let there be given $x, y \in X$. Assume first that $x \succsim y$. If $u(y) > u(x)$, by (ii) we have $y \succ x$, a contradiction to $x \succsim y$. Hence, $u(x) \geq u(y)$. Conversely, assume that $u(x) \geq u(y)$. If $x \succsim y$ didn't hold, we would have, by completeness, $y \succ x$, and then, applying (ii), $u(y) > u(x)$, a contradiction. Hence $u(x) \geq u(y)$ implies $x \succsim y$.

(i) implies (iii): Let there be given $x, y \in X$. Assume first that $x\tilde{}y$. Then $x \succsim y$ and $y \succsim x$. Applying (i) we have $u(x) \geq u(y)$ as well as $u(y) \geq u(x)$, hence $u(x) = u(y)$. Conversely, assume that $u(x) = u(y)$. Then we have $u(x) \geq u(y)$, which implies (by (i)) $x \succsim y$, as well as $u(y) \geq u(x)$ which implies $y \succsim x$, and $x\tilde{}y$ follows. □

To see that (iii) is strictly weaker than (i) and (ii), take a representation u of a relation $\succsim$ with more than one equivalence class, and define $v = -u$. Such a v will still represent indifferences as in (iii) but not preferences as in (i) or (ii).

2.2 Characterization theorems for maximization of utility

Assume that $\succsim$ is a binary relation on X of alternatives as above.

Theorem 5 *If X is finite or countably infinite, the following are equivalent:*

(i) $\succsim$ is complete and transitive

(ii) there is a function $u : X \to \mathbb{R}$ that represents $\succsim$.

A relation that is complete and transitive is called a *weak order*.

Proof: It is easy to see that (ii) implies (i), independently of the cardinality of X. Indeed, if u represents $\succsim$, the latter is complete because $\geq$ is complete on the real numbers, and $\succsim$ is transitive because so is $\geq$ (on the real numbers).

The main part of the proof is to show that (i) implies (ii). To this end, we can assume, without loss of generality, that the equivalence classes of $\tilde{}$ are singletons, that is, that no two distinct alternatives are equivalent. The reason is that from each equivalence class of $\tilde{}$, say A, we can choose a single representative $x_A \in A$. Clearly, if we restrict attention to $\succsim$ on $\{ x_A \mid A \in X/\tilde{} \}$ (where $X/\tilde{}$ denotes the quotient set, consisting of equivalent classes of $\tilde{}$), the relation is still complete and transitive and the cardinality of $X/\tilde{}$ is finite or countably infinite. Thus, is we manage to prove that (ii) implies (i) in this restricted case (of singleton equivalence classes), we will have a function $u : \{ x_A \mid A \in X/\tilde{} \} \to \mathbb{R}$ that represents $\succsim$ on $\{ x_A \mid A \in X/\tilde{} \}$. It remains to extend it to all of X in the obvious way (that respects equivalence, that is, set $u(y) = u(x_A)$ for all $y \in A$ and for every $A \in X/\tilde{}$).

(As we will shortly see, assuming that the equivalence sets are singletons doesn't make a huge difference. However, it's a good exercise to go over this reasoning if it's not immediately obvious.)

So let us now turn to the proof that (i) implies (ii) when the equivalence classes of $\tilde{}$ are singletons. Let us start with a simple proof for the finite case: assume that (i) holds, that X is finite, and define, for $x \in X$,

$$u(x) = \#\{\, y \mid x \succsim y \,\}$$

that is, the utility of an alternative x is simply the number of elements in X that x is at least as good as. This is like having any two alternatives compete, and count how many "victories" an alternative has. (This may remind you of sports tournaments.)

Clearly, u is a well-defined function. Also, because of transitivity, $x \succsim y$ implies $u(x) \geq u(y)$. Moreover, if $x \succ y$, then $x \in \{\, z \mid x \succsim z \,\}$ but $y \notin \{\, z \mid x \succsim z \,\}$ so that the inequality is strict, that is, $u(x) > u(y)$. Hence we have proved that $x \succsim y$ *iff* $u(x) \geq u(y)$.

It is easy to see that this proof doesn't extend to the infinite case (even if the set is countable), because the sets $\{\, y \mid x \succsim y \,\}$ may well be infinite, and then their cardinalities $\#\{\, y \mid x \succsim y \,\}$ are not real numbers. Moreover, even if we allowed the utility to take values in the extended reals, including ∞ and $-\infty$, we will not get a representation. For example, if we consider the rational numbers with the standard $\geq$ relation, all of then would have the same u value of ∞.

However, one can modify the proof a bit so that it will extend to the infinite (countable) case: defining

$$u(x) = \#\{\, y \mid x \succsim y \,\}$$

we basically counted, for each x, how many y's does it "beat". This is as if an alternative x collects "points" for its "victories" in the matches with other alternatives, and we assume that the points for all alternatives are equal. However, we could do the same trick with points that are not necessarily equal. Suppose that, for each y there is a "weight" $\alpha_y > 0$. Then, defining

$$u(x) = \sum_{\{y \mid x \succsim y\}} \alpha_y$$

– *if* this is a real number – the proof above goes through: transitivity proves that $x \succsim y$ implies $u(x) \geq u(y)$, and, because $\alpha_x > 0$, $x \succ y$ implies $u(x) > u(y)$. All that is left is to choose weights $\alpha_y > 0$ such that the summation above is always finite. This, however, can easily be done because X is countable. We can take any enumeration of X, $X = \{x_1, ..., x_n, ...\}$ and set $\alpha_{x_n} = \frac{1}{2^n}$. Since the entire series $\sum_n \alpha_{x_n}$ converges, $u(x)$ is well-defined, that it, it is a real number for every x.

Let us now look at a second proof, which uses induction. We consider a enumeration of X, $X = \{x_1, ..., x_n, ...\}$. Let $X_n = \{x_1, ..., x_n\}$ be the set consisting of the first n elements of X according to this enumeration. Clearly, $X_n \subset X_{n+1}$ for all $n \geq 1$ and $X = \cup_{n \geq 1} X_n$. We define u by induction: set $u(x_1) = 0$ and then, for each $n \geq 1$, we are about to define $u(x_{n+1}) \in \mathbb{R}$ given the definition of u on X_n. We will prove that, according to this definition, for every $n \geq 1$, if u represents $\succsim$ on X_n, it will also represent $\succsim$ on X_{n+1}, and then observe that this means also that u represents $\succsim$ on X.

Observe that, when we say "u represents $\succsim$ on X_n" we refer to the values of u on X_n, which are the first n numbers defined in the proof. Formally speaking, the function u on X_n is a different function that the function u defined on X_{n+1}. However, at stage $n+1$ of the proof we only define $u(x_{n+1})$ without changing the values of u on X_n, and thus there is no need to use a different notation for the function defined on the smaller set, X_n, and for its extension to the larger set, X_{n+1}.

The induction step, is, however, trivial: given X_n and u that is defined on it, let there be given x_{n+1}. If $x_{n+1} \prec x_i$ for all $i \leq n$, set $u(x_{n+1}) = \min_{i \leq n} u(x_i) - 1$. Symmetrically, if $x_{n+1} \succ x_i$ for all $i \leq n$, set $u(x_{n+1}) = \max_{i \leq n} u(x_i) + 1$. Otherwise, x_{n+1} is "between" two alternatives x_i, x_j that is, there are $i, j \leq n$

such that

$$x_i \prec x_{n+1} \prec x_j$$

and, for every $k \leq n$, $x_k \succeq x_j$ or $x_k \precsim x_j$. Setting

$$u(x_{n+1}) = \frac{1}{2}(u(x_i) + u(x_j))$$

completes the induction step.

Finally, it remains to be noted that this inductive process defines u over all of X. It is very important here that in the induction step we do not re-define the value of u defined in previous steps, so that u on X is well-defined. In fact, when we recall that functions are sets of ordered pairs, the function u on X is simply the union of the functions u defined over X_n – when we take the union over all n. To see this u represents $\succsim$ over all of X, consider two elements $x, y \in X$. They appear in the enumeration, say, $x = x_k$ and $y = x_l$. Then, taking $n = \max(k, l)$, we have $x, y \in X_n$, and then $x \succsim y$ iff $u(x) \geq u(y)$ because u represents $\succsim$ over X_n. $\square$

To see that the theorem, as stated, cannot generally be true if X is uncountable, consider Debreu's famous example of a lexicographic order (Debreu, 1959): let $X = [0,1]^2$ and $(x_1, x_2) \succsim_L (y_1, y_2)$ if and only if $[x_1 > y_1$ or $(x_1 = y_1$ and $x_2 \geq y_2)]$.

Proposition 6 *There is no function* $u : [0,1]^2 \to \mathbb{R}$ *that represents* $\succsim_L$.

Proof: If there were such a function, then, for every value of $x_1 \in [0,1]$ there would be an open interval of utility values,

$$I(x_1) = (u(x_1, 0), u(x_1, 1))$$

(with $u(x_1, 1) > u(x_1, 0)$) such that, for $x_1 > y_1$, $I(x_1)$ and $I(y_1)$ are disjoint (because $u(x_1, 0) > u(y_1, 1)$). However, on the real line we can only have countably many disjoint open intervals (for instance, because each such interval contains a rational number). $\square$

This means that in order to get a representation of $\succsim$ by a utility function when X is not countable we need to make additional assumptions.

One direction to follow (again, Debreu, 1959) is to assume that the set of alternatives X is a topological space, and require that $\succsim$ be continuous with respect to this topology. For example, in the case $X = \mathbb{R}^k$, we define continuity as follows: $\succsim$ is *continuous* if, for all $x, y \in X$ and every $\{x_n\} \subset \mathbb{R}^k$ such that $x_n \to x$, (i) if $x_n \succsim y$ for all n, then $x \succsim y$, and (ii) if $y \succsim x_n$ for all n, then $y \succsim x$.

Remark 7 *Assume that $X = \mathbb{R}^k$. Then $\succsim$ is continuous iff, for every $y \in \mathbb{R}^k$, the sets $\{x \in X \,|\, x \succ y\}$ and $\{x \in X \,|\, y \succ x\}$ are open (in the standard topology on $X = \mathbb{R}^k$).*

Given that definition, one may state:

Theorem 8 *If $X = \mathbb{R}^k$, the following are equivalent:*
(i) $\succsim$ is complete, transitive, and continuous
(ii) there is a continuous function $u : X \to \mathbb{R}$ that represents $\succsim$.

We will not prove this theorem here. Rather, we follow the other direction, due to Cantor (1915), where no additional assumptions are made on X. In particular, it need not be a topological space, and if it happens to be one, we still will not insist on continuity of u. Instead, Cantor used the notion of separability: $\succsim$ is *separable* if there exists a countable set $Z \subset X$ such that, for every $x, y \in X \backslash Z$, if $x \succ y$, then there exists $z \in Z$ such that $x \succsim z \succsim y$.

Theorem 9 *(For every X) The following are equivalent:*
(i) $\succsim$ is complete, transitive, and separable
(ii) there is a function $u : X \to \mathbb{R}$ that represents $\succsim$.

Let us now briefly understand the mathematical content of separability. The condition states, roughly speaking, that countably many element (those of Z) tell the entire story. If we want to know how an alternative x ranks, it suffices to know how it ranks relative to all elements of Z. Observe that, if X is countable, separability vacuously holds, since one may choose $Z = X$.

To see that the separability axiom has some mathematical power, and that it may give us hope for utility representation, let us see why it rules out Debreu's example above. In this example, suppose that $Z \subset X = [0,1]^2$ is a countable set. Consider its projection on the first coordinate, that is,

$$Z_1 = \{\, x_1 \in [0,1] \,|\, \exists x_2 \in [0,1] \quad s.t. (x_1, x_2) \in Z \} \,.$$

Clearly, Z_1 is also countable. Consider $x_1 \in [0,1] \backslash Z_1$ and note that $(x_1, 1) \succ (x_1, 0)$. However, no element of Z can be between these two in terms of preference, since it cannot have x_1 as its first coordinate.

Proof of Theorem 9

As above, the discussion is simplified if we assume, without loss of generality, that there are no equivalences.

We may go back to the two proofs of Theorem 5 and try to build on these ideas. The second proof, by which the values of $u(x_n)$ were defined by induction on n, doesn't generalize very graciously. For a countable set, one can have a enumeration of the elements, such that each one of them has only finitely many predecessors. This allowed us to find a value for $u(x_n)$ for each n, so that u represented $\succsim$ on the elements up to x_n. However, when X is uncountable, no such enumeration exists. Thus, there will be (many) elements x of X that have infinitely many predecessors. And then it might be impossible to find a value $u(x)$ that allows u to represent $\succsim$ on all the elements up to x. (For example, assume that $x \succ z$ and $y_n \succ x$ where we have already assigned the values $u(z) = 0$ and $u(y_n) = \frac{1}{n}$.)

However, the first proof does extend to the general set-up. Recall that, in the countable case, we agreed that for each y there would be a "weight" $\alpha_y > 0$ and that, given these weights, we would define

$$u(x) = \sum_{\{\, y \,|\, x \succsim y \,\}} \alpha_y.$$

You might think that, when X is uncountable, the corresponding idea would be to have an integral (over all $\{\, y \,|\, x \succsim y \}$ for each x) instead of a sum. But this would require a definition of an algebra on X (which is not the hard part) and a definition of $\alpha_y > 0$ so that the function $\alpha_\cdot$ is integrable relative to that algebra (which is harder). However, these complications are not necessary: the separability requirement says that countably many elements "tell the entire story". Hence, we should take the sum not over all $\{ y \mid x \succsim y \}$, but only over those elements of Z that are in this set.

Hence, for the proof that (i) implies (ii), let $Z = \{z_1, z_2, ...\}$ and define

$$u(x) = \sum_{z_i \in X, x \succsim z_i} \frac{1}{2^i} - \sum_{z_i \in X, z_i \succ x} \frac{1}{2^i}. \tag{2}$$

Clearly $u(x) \in \mathbb{R}$ for all x (in fact, $u(x) \in [-1, 1]$). It is easy to see that $x \succsim y$ implies $u(x) \geq u(y)$. To see the converse, assume that $x \succ y$. If one of $\{x, y\}$ is in Z, $u(x) > u(y)$ follows from the definition (2). Otherwise, invoke separability to find $z \in Z$ such that $x \succ z \succ y$ and then use (2).

Another little surprise in this theorem, somewhat less pleasant, is how messy the proof of the converse direction is. Normally we expect axiomatizations to have sufficiency, which is a challenge to prove, and necessity which is simple. If it is obvious that the axioms are necessary, they are probably quite transparent and compelling. (If, by contrast, sufficiency is hard to prove, the theorem is surprising in a good sense: the axioms take us a long way.) Yet, we should be ready to sometimes work harder to prove the necessity of conditions such as continuity, separability, and others that bridge the gap between the finite and the infinite.

In our case, if we have a representation by a function u, and if the range of u were the entire line $\mathbb{R}$, we would know what to do: to select a set Z that satisfies separability, take the rational numbers $\mathbb{Q} = \{q_1, q_2, ...\}$, for each such number q_i select z_i such that $u(z_i) = q_i$, and then show that Z separates X. The problem is that we may not find such a z_i for each q_i. In fact, it is even possible that $range(u) \cap \mathbb{Q} = \varnothing$.

In some sense, we need not worry too much if a certain q_i is not in the range of u. In fact, life is a little easier: if no element will have this value, we will not be asked to separate between x and y whose utility is this value. However, what happens if we have values of u very close to q_i on both sides, but q_i is missing? In this case, if we fail to choose elements with u-values close to q_i, we may later be confronted with $x \succ y$ such that $u(x) \in (q_i, q_i + \varepsilon)$ and $u(y) \in (q_i - \varepsilon, q_i)$ and we will not have an element of z with $x \succ z \succ y$.

Now that we see what the problem is, we can also find a solution: for each q_i, find a countable non-increasing sequence $\{z_i^k\}_k$ such that

$$\{u(z_i^k)\}_k \searrow \inf\{u(x)|u(x) > q_i\}$$

and a countable non-decreasing sequence $\{w_i^k\}_k$ such that

$$\{u(z_i^k)\}_k \nearrow \sup\{u(x)|u(x) < q_i\}$$

assuming that the sets on the right hand sides are non empty. The (countable) union of these countable sets will do the job. $\square$

In all of these representation results (Theorems 5, 8, 9), the function u is unique only up to increasing transformations. That is, if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, and u represents $\succsim$, then

$$v = f(u)$$

also represents $\succsim$. (In Theorem 8 one would need to require that f be continuous to guarantee that v is continuous on X, as is u. In the other two theorems we don't have a topology on X, and continuity of u or v is not defined. Therefore it makes no difference if f is continuous or not.) For that reason, a utility function u that represent $\succsim$ is called *ordinal*. Importantly, when we say that "u is ordinal" we don't refer to a property of the function u as a mathematical object per se, but to the way we use it. Saying that u is ordinal is like saying "I'm using the function u, but don't take me too literally; it's actually but an example of a function, a representative of a large class of functions that are equivalent in terms of their observable content, and any monotone transformation of u can be used as "the" utility function just as well. I'll try not to say anything that depends on the particular function u and that would not hold true if I were to replace u by a monotone transformation thereof, v."

3 Semi-Orders

3.1 Just Noticeable Difference

Proposition 3 showed that transitivity of $\succsim$ implies that of $\sim$. However, many alternatives involve variables that can be thought of as continuous, and in these cases it does not make sense to assume that $\sim$ is transitive due to our limited capacity to discern difference. In Luce's coffee mug example, a decision maker has preferences over coffee mugs with n grains of sugar. Not being able to discern a mug with n grains of sugar from one with $n+1$ grains, one can hardly expect there to be strict preference. Thus, we get indifference $\sim$ between any two consecutive alternatives, but this doesn't mean that we'll get indifference between any pair of alternatives.

Indeed, Weber's Law in psychophysiology (dating back to 1834), states that a person's ability to discern difference between perceptual stimuli is limited. He defined the just noticeable difference to be the minimal increase in a stimulus that is needed for the difference to be noticed. More precisely, if S is a level of a stimulus, let ΔS be the minimal quantity such that $(S + \Delta S)$ can be identified as larger than S with probability of at least 75%. Weber's Law states that $\Delta S/S$ is a constant, independent of S, say, $\lambda > 0$. In other words, if $S' > S$, the person will be able to tell that this is indeed the case (with probability of 75% or more) iff

$$S' > S + \Delta S = (1 + \lambda)\, S$$

or

$$\frac{S'}{S} > 1 + \lambda$$

or

$$\log(S') - \log(S) > \delta \equiv \log(1 + \lambda) > 0.$$

Inspired by this law, Luce (1956) was interested in strict preferences P that can be described by a utility function u through the equivalence

$$xPy \qquad \text{iff} \qquad u(x) - u(y) > \delta > 0 \qquad \forall x, y \in X. \tag{3}$$

If (3) for $u : X \to \mathbb{R}$ and $\delta > 0$, we say that the pair (u, δ) *L-represents* P.

Seeking to axiomatize L-representations, Luce considered the binary relation P (on a set of alternatives X) as primitive. The relation P is interpreted as strict preference, where $I = (P \cup P^{-1})^c$ – as absence of preference in either direction, or "indifference".[1]

Luce formulated three axioms, which are readily seen to be necessary for an L-representation. He defined a relation P to be a *semi-order* if it satisfied these three axioms, and showed that, if X is finite, the axioms are also sufficient for L-representation.

To state the axioms, it will be useful to have a notion of concatenation of relations. Given two binary relations B_1,$B_2 \subset X \times X$, let $B_1 B_2 \subset X \times X$ be defined as follows: for all $x, y \in X$,

$$xB_1B_2y \qquad \text{iff} \qquad \exists z \in X,\ xB_1z,\ zB_1y.$$

Observe that, if you think of a relation B as a binary matrix, whose rows and columns are members of X, and such that $B_{xy} = 1$ iff xBy (and $B_{xy} = 0$ otherwise), then the concatenated relation B_1B_2 precisely corresponds to the "product" of the matrices B_1 and B_2 is by "addition" we mean "or" (that is, $1 + 0 = 0 + 1 = 1 + 1 = 1$).

We can finally state Luce's axioms. The relation P (or (P, I)) is a *semi-order* if:

L1. P is irreflexive (that is, xPx for no $x \in X$);

L2. $PIP \subset P$

L3. $PPI \subset P$.

The meaning of L2 is, therefore: assume that $x, z, w, y \in X$ are such that $xPzIwPy$. Then it has to be the case that xPy. Similarly, L3 requires that xPy will hold whenever there are $z, w \in X$ such that $xPzPwIy$.

Since I is reflexive, each of L2, L3 implies transitivity of P (but not of I!). But L2 and L3 require something beyond transitivity of P. For example, if

[1] For a relation $P \subset X \times X$, define P^{-1} to be the inverse, i.e., $P^{-1} = \{(x, y)|(y, x) \in P\}$. Thus, $I = (P \cup P^{-1})^c$ is equivalent to say that xIy if and only if neither xPy nor yPx.

$X = \mathbb{R}^2$ and P is defined by Pareto domination, P is transitive but you can verify that it satisfies neither L2 nor L3.

Conditions L2 and L3 restrict the indifference relation I. For the Pareto relation P, the absence of preference, I, means intrinsic incomparability. Hence we can have, say, $xPzPwIy$ without being able to say much on the comparison between x and y. It is possible that y is incomparable to any of x, z, w because one of y's coordinates is higher than the corresponding coordinate for all of x, z, w. This is not the case if I only reflects the inability to discern small differences. Thus, L2 and L3 can be viewed as saying that the incomparability of alternatives, reflected in I, can only be attributed to issues os discernibility, and not to fundamental problems as in the case of Pareto dominance.

Looking at Luce's three conditions, you may wonder why not require also

L4. $IPP \subset P$.

The answer is that it follows from the previous two. More precisely

Proposition 10 *Assume that P is irreflexive. Then*

(i) L3 implies L4

(ii) L4 implies L3

(iii) L3 (and L4) do not imply L2

(iv) L2 does not imply L3 (or L4).

Proof: (i) Assume L3. To see that L4 holds, let there be given $x, y, z, w \in X$ such that $xIyPzPw$. We need to show that xPw. If not, we have either wPx or wIx. We argue that in either case, yPx. Indeed, if wPx, we have $yPzPwPx$. Recall that P is transitive by L3. Hence yPx. If, however, wIx, we have $yPzPwIx$ and, by L3, yPx. However, this is a contradiction because xIy.

(ii) Assume L4. To see that L3 holds, let there be given $x, y, z, w \in X$ such that $xPyPzIw$. We need to show that xPw. If not, we have either wPx or wIx. We argue that in either case, wPz. Indeed, if wPx, we have $wPxPyPz$ and (since L4 also implies transitivity of P), wPz. If, however, wIx, then $wIxPyPz$ and L4 implies wPz. Thus, in both cases we obtain wPz, which contradicts zIw.

(iii) Consider $X = \{x, y, z, w\}$ and $P = \{(x, y), (z, w)\}$. L3 and L4 hold vacuously (as there are no chains of two P relations) but L2 doesn't. (If it did, we should have xPw because $xPyIzPw$ – and, indeed, also zPy because $zPwIxPy$.)

(iv) Consider $X = \{x, y, z, w\}$ and $P = \{(x, y), (y, z), (x, z)\}$. L2 holds, as $PIP = \{(x, z)\}$ (because $xPyIyPz$ but there is no other quadruple of elements satisfying this chain of relations) and, indeed, $(x, z) \in P$. However, L3 does not hold (if it did, $xPyPzIw$ would imply xPw) nor does L4 (if it did, $wIxPyPz$ would imply wPz). $\square$

It is an easy exercise to show that L2 and L3 are necessary conditions for an L-representation to exist. It takes more work to prove the following.

Theorem 11 *(Luce) Let X be finite. $P \subset X \times X$ is a semi-order if and only if there exists $u : X \to \mathbb{R}$ that L-represents it.*

We will not prove this theorem here, but we will make several comments about it.

If we drop L3 (but do not add L4), we get a family of relations that Fishburn (1970b, 1985) defined as *interval relations*. Fishburn proved that, if X is finite, a relation is an interval relation if and only if it can be represented as follows: for every $x \in X$ we have an interval, $(b(x), e(x))$, with $b(x) \le e(x)$, such that

$$xPy \qquad \text{iff} \qquad b(x) > e(y) \qquad \forall x, y \in X$$

that is, xPy iff the entire range of values associated with x, $(b(x), e(x))$, is higher than the range of values associated with y, $(b(y), e(y))$.

Given such a representation, you can define $u(x) = b(x)$ and $\delta(x) = e(x) - b(x)$ to get an equivalent representation

$$xPy \qquad \text{iff} \qquad u(x) - u(y) > \delta(y) \qquad \forall x, y \in X \tag{4}$$

Comparing (4) to (3), you can think of (4) as a representation with a variable just noticeable difference, whereas (3) has a constant jnd, which is normalized to 1.

3.2 A note on the proof

If P is a semi-order, one can define from it a relation $Q = PI \cup IP$. That is, xQy if there exists a z such that (xPz and zIy) or (xIz and zPy). This is an indirectly revealed preference: suppose that xIy but xPz and zIy. This means that a direct comparison of x and y does not reveal a noticeable difference, and therefore no preference either. But, when comparing x and y to another alternative z, it turns out that x is different enough from z to be preferred to it, while y isn't. Indirectly, we find evidence that x is actually better than y for our decision maker, even though the decision maker herself cannot discern the difference when the two are presented to her. (Similar logic applies if $xIPy$.)

If P is a semi-order, Q turns out to be the strict part of a weak order:

Claim 12 *Q is transitive*

Proof: Assume xQy and yQz.

If $xPwIy$ and $yPtIz$ then $xPIPt$ hence xPt and tIz that is $xPIz$.

If $xIwPy$ and $yPtIz$ then $xIPPt$ hence xPt and tIz that is $xPIz$.

If $xIwPy$ and $yItPz$ then xIw and $wPIPz$ hence xIw and wPz hence $xIPz$.

If $xPwIy$ and $yItPz$ then we know that tPx does not hold. If it did, $tPxPwIy$ that is, $tPPIy$ and tPy would follow (while we know that yIt). Hence xPt, in which case $xPtPz$, and xPz and xQz follow, or xIt, but then $xItPz$, that is $xIPz$. $\square$

The equivalence relation that corresponds to the relation Q is denoted by E. Formally, define $E = (Q \cup Q^{-1})^c$.

Claim 13 *E is transitive*

Proof: Define ˜ as follows: $x\tilde{}y$ if for every z, ($xPz \Leftrightarrow yPz$) and ($zPx \Leftrightarrow zPy$). Clearly, ˜ is an equivalence relation. Also, $x\tilde{}y$ implies xEy. To see the converse, assume that xEy. If there exists z such that $xPzPy$ (or $yPzPx$) then xPy (yPx) and $x\tilde{}y$ cannot hold. Hence $xPz \Rightarrow yPz$ and vice versa. Similarly, $zPx \Rightarrow zPy$ and vice versa. $\square$

Moreover, one can get an L-representation of P by a function u that simultaneously also satisfies

$$xQy \qquad \text{iff} \qquad u(x) - u(y) > 0 \qquad \forall x, y \in X. \tag{5}$$

3.3 Uniqueness of the utility function

We finally get back to the question of uniqueness. How unique is a utility function that L-represents a semi-order P? In general the answer may be very messy. Let us therefore suppose that we are dealing with a large X so that we can think of the range of the utility function being all of $\mathbb{R}$, or at least an open interval in $\mathbb{R}$.

For this to be the case, one would need additional assumptions on P, beyond it being a semi-order: first, if we wish to have a representation of P by (3) and, simultaneously, of Q by (5), we'd need something like separability to guarantee that Q is separable. On top of that, there would be assumptions that are specific to semi-orders. For example, if $x_1 P x_2 P x_3 P ...$ and $x_n P y$ for every n, we won't be able to have a representation of P by (3). Thus, we would need to requires that any infinite P-chain cannot be P-bounded from above or from below. We won't get into the characterization here, and will simply assume that P is such that (3) holds with $range(u) = \mathbb{R}$.

How unique are (u, δ), then? Clearly, if we shift u by a constant, defining $v(x) = u(x) + c$, the representation does not change. Similarly, if we multiply both u and δ by a positive constant, the representation is unaffected. Hence, when we may assume, without loss of generality, that we normalize (u, δ) so that $\delta = 1$ and, for some $x_0 \in X$, $u(x_0) = 0$. Assume now that v is also such a function that satisfies (3) and (5). Thus, $u, v : X \to \mathbb{R}$ satisfy, for all $x, y \in X$,

$$\begin{aligned} xPy \quad &\Leftrightarrow \quad u(x) - u(y) > 1 \;\Leftrightarrow\; v(x) - v(y) > 1 \\ xQy \quad &\Leftrightarrow \quad u(x) - u(y) > 0 \;\Leftrightarrow\; v(x) - v(y) > 0 \end{aligned}$$

with $u(x_0) = v(x_0) = 0$ and $range(u) = range(v) = \mathbb{R}$.

The functions u and v can be quite different on $[0,1]$. But if x is such that $u(x) = 1$, we will also have to have $v(x) = 1$. To see this, imagine that $v(x) > 1$. Then there are alternatives y with $v(y) \in (1, v(x))$. This would mean that, according to v, $yP0$, while according to u, $yI0$, a contradiction.

The same logic applies to any point we start out with. That is, for every x, y,

$$u(x) - u(y) = 1 \iff v(x) - v(y) = 1$$

and this obviously generalizes to $(u(x) - u(y) = k \Leftrightarrow v(x) - v(y) = k)$ for every $k \in \mathbb{Z}$. Moreover, we obtain

$$|u(x) - v(x)| < 1 \qquad \forall x.$$

In other words, the just noticeable difference, which we here normalized to 1, has an observable meaning. Every function that L-represents P has to have the same jnd. Similarly, if we consider any two alternatives x and y and find that

$$3 < u(x) - u(y) < 4$$

we know that these inequalities will also hold for any other utility function v. And the reason is, again, that utility differences became observable to a certain degree: we have an observable distinction between "greater than the jnd" and "smaller than (or equal to) the jnd". This distinction, however coarse, gives us some observable anchor by which we can measure distances along the utility scale: we can count how many integer jnd steps exist between alternatives.

One can make a stronger claim if one recalls that the semi-orders were defined for a given probability threshold, say, $p = 75\%$. If one varies the probability, one can obtain a different semi-order. Thus we have a family of semi-orders $\{P_p\}_{p>.5}$. Under certain assumptions, all these semi-orders can be represented simultaneously by the same utility function u, and a corresponding family of jnd's, $\{\delta_p\}_{p>.5}$ such that

$$\begin{aligned} xP_py &\iff u(x) - u(y) > \delta_p \\ xQy &\iff u(x) - u(y) > 0. \end{aligned}$$

In this case, it is easy to see that the utility u will be unique to a larger degree than before. We may even find that, as $p \to .5$, $\delta_p \to 0$, that is, that if we are willing to make do with very low probabilities of detection, we will get very low jnd's, and correspondingly, any two functions u and v that L-represent the semi-orders $\{P_p\}_{p>.5}$ will be identical.

Observe that the uniqueness result depends discontinuously on the jnd δ: the smaller is δ, the less freedom we have in choosing the function u, since $\sup |u(x) - v(x)| \leq \delta$. But when we consider the case $\delta = 0$, we are back with a weak order, for which u is only ordinal.

4 Choice Functions

The binary relation approach assumes that we observe choices between pairs of alternatives. More generally, given a set of alternatives X, we may assume that the choice is observed within various subsets of X, and not only between pairs. Assume that X is finite, and denote by $\mathcal{B} \subset 2^X \backslash \{\varnothing\}$ the collection of (non-empty) subsets of X that are choice sets, that is, that choice within each of them can be observed. This choice is assumed to be a subset of the set offered. Thus we define a *choice correspondence* to be a function

$$C \quad : \quad \mathcal{B} \rightarrow 2^X \backslash \{\varnothing\}$$

$$\text{with}$$

$$C(B) \quad \subset \quad B \qquad \forall B \in \mathcal{B}.$$

Further, we assume that $\mathcal{B}$ includes all subsets of size ≤ 3, so that the choice functions we consider will be sufficiently informative.

In this context, we define the *Weak Axiom of Revealed Preference (WARP)* by:

WARP: If for some B, $x, y \in B \in \mathcal{B}$, $x \in C(B)$, then, for any $B' \in \mathcal{B}$ with $x, y \in B'$, $y \in C(B')$ implies $x \in C(B')$.

This axiom states that, if in one context (B), where y was available ($y \in B$), x was chosen ($x \in C(B)$), then in any other context (B') where both are available ($x, y \in B'$), if y is good enough to be chosen ($y \in C(B')$), then so is x ($x \in C(B')$). Thus, if in one instance x was observed to be at least as good as y, we will never find that y is strictly better than x.

A choice function that satisfies WARP can be thought of as a binary relation. To be precise, one may start with a choice function C, and, if it satisfies WARP, define a binary relation $\succsim^*$ such that C picks the $\succsim^*$-maximal elements in B for every $B \in \mathcal{B}$. Conversely, if one starts with a binary relation $\succsim$, one may define the choice function that selects the $\succsim$-maximal elements (n B for every $B \in \mathcal{B}$) and show that it satisfies WARP. Details follow.

Let us first assume that a choice function $C : \mathcal{B} \rightarrow 2^X \backslash \{\varnothing\}$ is given. Define a binary relation $\succsim^* = \succsim^* (C)$ as follows: for every $x, y \in X$, $x \succsim^* y$ if (and only if)[2] there *exists* $B \in \mathcal{B}$ such that $x, y \in B$, and $x \in C(B)$. That is, we say that $x \succsim^* y$ if there is a context in which x was revealed to be at least as desirable as y.

Taking $\succ^*$ to be the asymmetric part of $\succsim^*$, we find that $x \succ^* y$ iff (i) for at least one $B \in \mathcal{B}$ with $x, y \in B$, we have $x \in C(B)$ but (ii) for no $B \in \mathcal{B}$ such that $x, y \in B$, is it the case that $y \in C(B)$.

Note that, if then there exists a $B \in \mathcal{B}$ such that $x, y \in B$, and $x \in C(B)$ but not $y \in C(B)$. We could therefore say that x was "revealed to be strictly preferred to" y. Indeed, it makes sense to define this formally: we write $x \succ' y$, if there exists $B \in \mathcal{B}$, with $x, y \in B$, such that $x \in C(B)$ but $y \notin C(B)$. Thus, $x \succ^* y$ implies $x \succ' y$. But the converse isn't generally true: it is possible that, given one B only x is chosen, and given another, B', only y is chosen (while x, y are in both B and B'). That is, the definition of $\succ'$ allows for the possibility that $x \succ' y$ and $y \succ' x$. By contrast, the definition of $\succ^*$ implies asymmetry: if $x \succ^* y$, we know that in some contexts (sets $B \in \mathcal{B}$ with $x, y \in B$), x was chosen, but in *none* was y chosen.

Conversely, if we start with a binary relation $\succsim$, it makes sense to define a choice function by seeking the $\succsim$-maximal elements. Formally, for any $\succsim \subset X \times X$ we can define

$$C^*(B) = C^*(B, \succsim) = \{\, x \in B \,|\, x \succsim y \qquad \forall y \in B\}$$

(observe that this is not necessarily a choice function as we're not guaranteed that the set hereby defined is non-empty.)

We can now state formally the equivalence between binary orders that are complete and transitive and choice functions that satisfy WARP. Let us start with the more immediate result:

[2] In case we haven't mentioned this: definitions are always characterizations, that is, "if and only if" statements. For this very reason, it is considered better style not to write "... and only if" in definitions.

Proposition 14 *If $\succsim$ is a weak order, then $C^*(B, \succsim)$ is a choice function satisfies WARP. Furthermore, the relation corresponding to C^* is $\succsim$: $\succsim = \succsim^* (C^*)$.*

Proof: Recall that X is finite. Then $C^*(B) \neq \varnothing$ for every $B \in \mathcal{B}$. Moreover, $\succsim$ can be represented by a function u and $C^*(B)$ consists precisely of the u-maximizers in B. It follows that WARP holds: if for some B, $x, y \in B \in \mathcal{B}$, $x \in C(B)$, we know that $u(x) \geq u(y)$ and then, for any $B' \in \mathcal{B}$ with $x, y \in B'$, if $y \in C(B')$ then y is a maximizer of u over B' and then so is x, and this implies $x \in C(B')$.

As for the "furthermore" part, observe that, given C^*, the relation $\succsim^* = \succsim^* (C^*)$ is defined by $x \succsim^* y$ iff there exists $B \in \mathcal{B}$ such that $x, y \in B$, and $x \in C^*(B)$. If indeed such a B exists, we know that that $u(x) \geq u(y)$ and $x \succsim y$. Conversely, if $x \succsim y$ and $u(x) \geq u(y)$, then we have $x \in C^*(\{x, y\})$ and $x \succsim^* y$ holds. Hence $\succsim^* = \succsim$. $\square$

Conversely, let us now start with a choice function that satisfies WARP and define the relation from it.

Proposition 15 *If C satisfies WARP, then $\succsim^* = \succsim^* (C^*)$ is a weak order, it satisfies $C^*(B, \succsim^*) = C(B)$ for all $B \in \mathcal{B}$, and it is the unique weak order that satisfies this equation.*

Proof: Let there be given a choice function C that satisfies WARP. To see that $\succsim^* = \succsim^* (C^*)$ is complete, consider $B = \{x, y\}$ (which is in $\mathcal{B}$ as we assumed that all sets with no more than three elements are in $\mathcal{B}$). Because $C(\{x, y\}) \neq \varnothing$, it has to be the case that $x \in B$, and then $x \succsim^* y$, or $y \in B$, and then $y \succsim^* x$ (or both).

To see that $\succsim^*$ is transitive, assume that $x \succsim^* y$ and $y \succsim^* z$, and we will prove that $x \succsim^* z$. As $x \succsim^* y$, there exists $D \in \mathcal{B}$ such that $x, y \in B$, and $x \in C(D)$. WARP then implies that the same would hold for $D' = \{x, y\} \in \mathcal{B}$: $x \in C(\{x, y\})$. Similarly, $y \succsim^* z$ means that there exists some $E \in \mathcal{B}$ such that $y, z \in E$, and $y \in C(E)$ and this implies also $y \in C(\{y, z\})$. Let us now consider $B = \{x, y, z\} \in \mathcal{B}$. We need to show that $x \in C(\{x, y, z\})$ (and then, by definition of $\succsim^*$, $x \succsim^* z$ is established). Assume that this is not the case, that is, $x \notin C(\{x, y, z\})$. Can it be the case that $y \in C(\{x, y, z\})$? The negative answer is given by WARP: since $x \in C(\{x, y\})$, x will be chosen whenever y is (provided they are both available). Hence we find that $x \notin C(\{x, y, z\})$ implies $y \notin C(\{x, y, z\})$. But, by the same token, $y \notin C(\{x, y, z\})$ implies $z \notin C(\{x, y, z\})$ and it follows that, if $x \notin C(\{x, y, z\})$ then $C(\{x, y, z\}) = \varnothing$, a contradiction to the definition of choice functions. Hence $x \in C(\{x, y, z\})$ and $\succsim^*$ is transitive.

We now turn to show that, if we define C^* from the relation $\succsim^*$, we get the function C that we started out with. That is, we wish to show that, for every $B \in \mathcal{B}$, $C^*(B, \succsim^*) = C(B)$.

Fix B. To see that $C^*(B, \succsim^*) \subset C(B)$, let $x \in C^*(B, \succsim^*)$, that is, x is a $\succsim^*$-maximum in B. Choose $y \in C(B)$. Since x is a $\succsim^*$-maximum in B, we know that $x \succsim^* y$. By definition of $\succsim^*$, for some B', $x, y \in B'$, $x \in C(B')$. But then WARP implies $x \in C(B)$ and $C^*(B, \succsim^*) \subset C(B)$ is established.

To see the converse inclusion, namely, that, $C(B) \subset C^*(B, \succsim^*)$, let $x \in C(B)$. By definition of $\succsim^*$, this implies that $x \succsim^* y$ for every $y \in B$. That is, x is a $\succsim^*$-maximum in B. But this, in turn, is precisely the definition of $C^*(B, \succsim^*)$. Hence $x \in C^*(B, \succsim^*)$ and $C(B) \subset C^*(B, \succsim^*)$ also holds.

Finally, to see uniqueness of the relation $\succsim^*$, it suffices to consider the sets B's that are pairs, and to observe that C on these sets is sufficient to define $\succsim^*$ uniquely. $\square$

5 von Neumann-Morgenstern/Herstein-Milnor Theorem

In this section we present a theorem that is some combination of results, by people whose names are in the title. de Finetti was the first to indicate the type of result he needed to have, and we'll discuss the context of his result later on. von Neumann and Morgenstern (vNM) had the famous theorem which we will study later on. The theorem we present here is slightly more general than the result they proved, as it will be used for other structures as well. The generalized version suggested here is still a special case of the generalization of vNM's theorem provided by Herstein and Milnor (1953).

Let there be an underlying set A and suppose that we are interested in objects of choice that are described as real-valued functions on A. Thus,

$$X \subset \mathbb{R}^A.$$

The structure of linear functions allows us to define a *mixture operation*: for every $x, y \in X$ and every $\alpha \in [0,1]$ define $\alpha x + (1-\alpha)y \in \mathbb{R}^A$ is defined pointwise, that is, it is given by

$$(\alpha x + (1-\alpha)y)(i) = \alpha x(i) + (1-\alpha)y(i)$$

for every $i \in A$. We assume that X is a convex set, so that $\alpha x + (1-\alpha)y \in X$ for $x, y \in X$ and every $\alpha \in [0,1]$.

In de Finetti's set-up, the set A consists of states of the world, and $x \in X$ designates a "bet": an act that, given state $i \in A$, yields a payoff $x(i)$, which is assumed to be monetary. In this case one may assume that $X = \mathbb{R}^A$, or perhaps add some measurability constraints if A is infinite. In vNM's set-up, A denotes possible outcomes, and an element $x \in X$ is a lottery, obtaining the outcome i with probability $x(i)$. In this case attention is restricted to functions $x \in \mathbb{R}^A$ that are non-negative, assume positive values for only finitely many i's, and add up to 1, that is, an element $x \in X$ is a lottery on A, with a finite support. Additional examples will be discussed later on.

A relation $\succsim \subset X \times X$ will be assumed to satisfy the following three axioms:

A1. **Weak order**: $\succsim$ is complete and transitive.

A2. **Continuity**: For every $x, y, z \in X$, if $x \succ y \succ z$, there exist $\alpha, \beta \in (0,1)$ such that

$$\alpha x + (1-\alpha)z \succ y \succ \beta x + (1-\beta)z.$$

A3. **Independence**: For every $x, y, z \in X$, and every $\alpha \in (0,1)$,

$$x \succ y \qquad \textit{implies} \qquad \alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z.$$

A function $U : X \to \mathbb{R}$ is called *affine* if, for every $x, y \in X$, and every $\alpha \in [0,1]$,

$$U(\alpha x + (1-\alpha)y) = \alpha U(x) + (1-\alpha)U(y)$$

Theorem 16 *A relation $\succsim \subset X \times X$ satisfies A1-A3 if and only if it can be represented by an affine $U : X \to \mathbb{R}$.*

Furthermore, in this case, the function U is unique up to a positive affine transformation: an affine function $V : X \to \mathbb{R}$ also represents $\succsim$ iff there are $c > 0$ and $d \in \mathbb{R}$ such that

$$U(x) = cV(x) + d \qquad \forall x \in X.$$

Proof: We mention that necessity of the axioms is straightforward. Also, it is easy to see that, if U is an affine function that represents $\succsim$, so will be any increasing affine transformation thereof, V. The main part of the proof is the sufficiency of the axioms; along the proof of sufficiency we will develop tools that would make uniqueness easy to establish.

Let us assume, then, that A1-A3 hold. We start with a few lemmas.

Lemma 17 *For every $x, y \in X$, if $x \succ y$,* *(i) for every* $\lambda \in (0,1)$,

$$x \succ \lambda x + (1-\lambda)y \succ y$$

(ii) for every $\lambda, \mu \in (0,1)$ *with* $\lambda > \mu$,

$$x \succ \lambda x + (1-\lambda)y \succ \mu x + (1-\mu)y \succ y.$$

Proof: (i) Assume that $x \succ y$. Use A3 for $z = x$ and $\alpha = 1-\lambda$ to obtain

$$x = \alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z = \lambda x + (1-\lambda)y.$$

Then use A3 again, for $z = y$ and $\alpha = \lambda$ to obtain

$$\lambda x + (1-\lambda)y = \alpha x + (1-\alpha)y \succ \alpha y + (1-\alpha)z = y.$$

(ii) Assume that $x \succ y$ and $\lambda > \mu$. By (i) we know that $\lambda x + (1-\lambda)y \succ y$. Defining $x' = \lambda x + (1-\lambda)y$, apply (i) again for $\lambda' = \frac{\mu}{\lambda}$ to conclude that

$$x' \succ \lambda' x' + (1-\lambda')y \succ y$$

and, observing that

$$\lambda' x' + (1-\lambda')y = \frac{\mu}{\lambda}\lambda x + \frac{\mu}{\lambda}(1-\lambda)y + (1-\frac{\mu}{\lambda})y = \mu x + (1-\mu)y$$

we get $\lambda x + (1-\lambda)y \succ \mu x + (1-\mu)y$. $\square$

Lemma 18 *For every* $x, y \in X$*, if* $x \sim y$*, then, for every* $\lambda \in [0,1]$,

$$x \sim \lambda x + (1-\lambda)y \sim y$$

Proof: Let there be $x\tilde{}y$ and assume that for some $\lambda \in (0,1)$, $z \equiv \lambda x + (1-\lambda)y$ does not satisfy $z\tilde{}x$. Assume that $z \succ x, y$. (The proof for the case $z \prec x, y$ is symmetric.) By the previous lemma, we know that

$$z \succ \alpha z + (1-\alpha)x \succ x$$

for every $\alpha \in (0,1)$. Thus, for every $\mu > \lambda$,

$$z = \lambda x + (1-\lambda)y \succ \mu x + (1-\mu)y \succ x \sim y.$$

Next consider $\mu > \lambda$ and observe that $\mu x + (1-\mu)y \succ y$. Pick one such μ and denote $w = \mu x + (1-\mu)y$, so that $z \succ w \succ y$.

Since $w \succ y$, for every $\beta \in (0,1)$ we have

$$w \succ \beta w + (1-\beta)\, y \succ y$$

but for $\beta = \frac{\lambda}{\mu}$ we obtain $w \succ z$, a contradiction. Hence we have $z \sim x \sim y$. $\square$

Lemma 19 *For every $x, y, z \in X$, and every $\alpha \in (0,1)$,*

$$x \succsim y \qquad iff \qquad \alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z.$$

Proof: The independence axiom states that $x \succ y$ implies $\alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$. Hence we only need to prove that $x \sim y$ implies $\alpha x + (1-\alpha)z \sim \alpha y + (1-\alpha)z$ for every $x, y, z \in X$ and every $\alpha \in (0,1)$.

Let there be given x, y, z with $x \sim y$. If $z \sim x \sim y$, then, by Lemma 18, $\alpha x + (1-\alpha)z \sim x \sim y \sim \alpha y + (1-\alpha)z$. We are therefore interested in the case $x \sim y \succ z$ or $z \succ x \sim y$. Consider the first ($x \sim y \succ z$), as the second is proved in a symmetric way.

Observe that, for every $\delta < 1$, $y \succ \delta x + (1-\delta)z$ because $y \sim x$ and $x \succ \delta x + (1-\delta)z$. We claim that, for $1 \geq \beta > \gamma \geq 0$, $\beta y + (1-\beta)z \succ \gamma x + (1-\gamma)z$. To see this, consider $x' = (\gamma/\beta)\, x + (1-\gamma/\beta)\, z$ so that $y \succ x'$ and thus $\beta x + (1-\beta)z \succ \beta x' + (1-\beta)z = \gamma x + (1-\gamma)z$.

Assume that, contrary to our claim, for some $\alpha > 0$ we have $\alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$. (The case of the opposite preference is proven similarly.) Since $\alpha y + (1-\alpha)z \succ z$, continuity implies that for some μ

$$\mu\,(\alpha x + (1-\alpha)z) + (1-\mu)\, z \succ \alpha y + (1-\alpha)z$$

that is, that there exists $\nu < \alpha$ such that $\nu x + (1-\nu)z \succ \alpha y + (1-\alpha)z$. But this is in contradiction to

$$\beta > \gamma \qquad \Rightarrow \qquad \beta y + (1-\beta)z \succ \gamma x + (1-\gamma)z.$$

$\square$

The next lemma is a key step in defining the utility value for an alternative:

Lemma 20 *Assume that $x, y, z \in X$ are such that $x \succ y$ and $x \succsim z \succsim y$. Then there exists a unique $\alpha = \alpha(x, y, z) \in [0, 1]$ such that $z \sim \alpha x + (1 - \alpha) y$.*

Proof: If $z \sim x$ then $\alpha = 1$ satisfies the condition. It is unique because, for any $\alpha < 1$ we have $x \succ \alpha x + (1 - \alpha) y$. Similarly, $z \sim y$ implies that $\alpha = 0$ is the unique value that provides $z \sim \alpha x + (1 - \alpha) y$.

Assume, then, that $x \succ z \succ y$. Let

$$\begin{aligned} G &= \{ \alpha \in [0, 1] \mid \alpha x + (1 - \alpha) y \succ z \} \\ E &= \{ \alpha \in [0, 1] \mid \alpha x + (1 - \alpha) y \sim z \} \\ B &= \{ \alpha \in [0, 1] \mid \alpha x + (1 - \alpha) y \prec z \} \end{aligned}$$

By completeness, $\{G, E, B\}$ is a partition of $[0, 1]$. Since $x \succ z \succ y$, $1 \in G$ and $0 \in B$. Lemma 17 implies that G and B are convex, and that E cannot consist of more than one point (taking transitivity into account). We still need to show, however, that E is not empty.

Suppose that E were empty. Denote $\alpha^* = \inf G = \sup B$. Then, either

$$B = [0, \alpha^*] \qquad ; \qquad G = (\alpha^*, 1]$$

or

$$B = [0, \alpha^*) \qquad ; \qquad G = [\alpha^*, 1].$$

In either case, the Continuity axiom (A2) is violated: in the first,

$$x \succ z \succ \alpha^* x + (1 - \alpha^*) y$$

but any non-trivial mixture of x with $\alpha^* x + (1 - \alpha^*) y$, that is, any

$$\lambda x + (1 - \lambda) \, \alpha^* x + (1 - \alpha^*) y$$

for $\lambda \in (0, 1)$ is strictly preferred to z. In the second case,

$$\alpha^* x + (1 - \alpha^*) y \succ z \succ y$$

and z is preferred to any non-trivial mixture of $\alpha^* x + (1 - \alpha^*) y$ with y, that is, z is preferred to any

$$\lambda \left[\alpha^* x + (1 - \alpha^*) y \right] + (1 - \lambda) \, y$$

for $\lambda \in (0,1)$.

$$\alpha x + (1-\alpha)y \succ z$$

Thus, continuity necessitates that both B and G be open intervals in $[0,1]$. Since $[0,1]$ cannot be split into two disjoint open intervals, we find that $E \neq \varnothing$. $\square$

It will be useful to have a notation for the alternatives that are, in terms of preferences, in the range of a set of alternatives. For $Y \subset X$, define

$$[Y]_{\succsim} = \left\{ x \in X \;\middle|\; \begin{array}{ll} \exists g \in Y, & g \succsim x \\ \exists b \in Y, & x \succsim b \end{array} \right\}$$

(We will only use this notation for finite, and rather small sets Y. Still, this notation will save some lines.) For example, for two alternatives, $b, g \in X$ such that $g \succsim b$,

$$[\{b,g\}]_{\succsim} = \{ x \in X \mid g \succsim x \succsim b \}$$

In this case, we can also simplify notation and write $[b,g]_{\succsim}$ for $[\{b,g\}]_{\succsim}$. With this notation, we can state the following.

Lemma 21 *Let there be $b, g \in X$ such that $g \succ b$. There exists an affine $U_{b,g} : [b,g]_{\succsim} \to \mathbb{R}$ that represents $\succsim$ on $[b,g]_{\succsim}$. Moreover, it is unique up to a positive affine transformation.*

Proof: By Lemma 20, for every $x \in [b,g]_{\succsim}$ there is a unique $\alpha = \alpha(g,b,x) \in [0,1]$ such that $x \tilde{} \alpha g + (1-\alpha) b$. Define $U(x) = \alpha(g,b,x)$.

To see that U represents $\succsim$, consider $x, y \in [b,g]_{\succsim}$. We have

$$x \tilde{} \alpha(g,b,x) g + (1-\alpha(g,b,x)) b$$

$$y \tilde{} \alpha(g,b,y) g + (1-\alpha(g,b,y)) b$$

and thus

$$\begin{array}{rcl} x & \succsim & y \qquad iff \\ \alpha(g,b,x) g + (1-\alpha(g,b,x)) b & \succsim & \alpha(g,b,y) g + (1-\alpha(g,b,y)) b \end{array}$$

which, in light of Lemma 17, is equivalent to

$$\alpha(g,b,x) \geq \alpha(g,b,y)$$

or to

$$U(x) \geq U(y).$$

Next, to see that U is affine, consider $z = \lambda x + (1-\lambda) y$ for $x, y \in [b,g]_{\succsim}$. Observe that, by Lemmas 17 and 18, $z \in [b,g]_{\succsim}$. From

$$x \tilde{} \alpha(g,b,x) g + (1-\alpha(g,b,x)) b$$

and Lemma 19 we deduce

$$\begin{aligned} z &= \lambda x + (1-\lambda) y \\ &\tilde{} \lambda [\alpha(g,b,x) g + (1-\alpha(g,b,x)) b] + (1-\lambda) y \end{aligned} \tag{6}$$

and from

$$y \tilde{} \alpha(g,b,y) g + (1-\alpha(g,b,y)) b$$

we similarly obtain that (6) is also equivalent to

$$\begin{aligned} & \lambda [\alpha(g,b,x) g + (1-\alpha(g,b,x)) b] \\ & + (1-\lambda) [\alpha(g,b,y) g + (1-\alpha(g,b,y)) b] \\ = \; & [\lambda \alpha(g,b,x) + (1-\lambda) \alpha(g,b,y)] g \\ & + [1 - [\lambda \alpha(g,b,x) + (1-\lambda) \alpha(g,b,y)]] b. \end{aligned}$$

Thus

$$\begin{aligned} U_{b,g}(z) &= \lambda \alpha(g,b,x) + (1-\lambda) \alpha(g,b,y) \\ &= \lambda U_{b,g}(x) + (1-\lambda) U_{b,g}(y). \end{aligned}$$

Finally, we wish to prove that this U is unique. Assume that $V : [b,g]_{\succsim} \to \mathbb{R}$ is also affine and represents $\succsim$. Define

$$\begin{aligned} c &= V(g) - V(b) > 0 \\ d &= V(b) \end{aligned}$$

so that

$$V(x) = cU_{b,g}(x) + d$$

for $x = b, g$. For any other $x \in [b, g]_{\succsim}$, recall that

$$x \tilde{} \alpha(g, b, x) g + (1 - \alpha(g, b, x)) b$$

and, because V represents $\succsim$,

$$\begin{aligned} V(x) &= \alpha(g, b, x) V(g) + (1 - \alpha(g, b, x)) V(b) \\ &= \alpha(g, b, x)(c + d) + (1 - \alpha(g, b, x)) d \\ &= U_{b,g}(x)(c + d) + (1 - U_{b,g}(x)) d \\ &= cU_{b,g}(x) + d. \end{aligned}$$

which completes the proof of the Lemma. □

Clearly, we're nearing the end of the proof. We have more or less what we needed: an affine function that represents preferences. This function can be defined over each preference interval separately, no matter how large it is. Thus, if X happens to have a maximal and a minimal elements, we're done: we only need to apply Lemma 21 to the interval between the minimal and the maximal element, which spans all of X. However, some more work is needed if maximal or minimal elements fail to exist.

We define the function U as follows. If all elements in X are equivalent, we set $U(x) = 0$ for all x. This function is affine, and it represents preferences. Moreover, it is unique up to a positive affine transformation: any other function that represents preferences has to be a constant as well. Otherwise, not all elements of X are equivalent. Thus, there are $b, g \in X$ such that $g \succ b$. Fix these two alternatives until the end of the proof, and set $U(b) = 0$ and $U(g) = 1$. For $x \neq b, g$, define $U(x)$ as follows:

(i) for $x \in [b, g]$, define $U(x) = U_{b,g}(x)$;

(ii) for $x \succ g$, define $U(x) = 1/U_{b,x}(g)$ so that

$$\begin{aligned}(0,1,U(x)) &= (U_{b,g}(b), U_{b,g}(g), U(x)) \\ &= c(U_{b,x}(b), U_{b,x}(g), U_{b,x}(x)) \\ &= c(0, U_{b,x}(g), 1)\end{aligned}$$

for $c = 1/U_{b,x}(g) > 0$;

(iii) for $x \prec b$, define $U(x) = -\frac{U_{x,g}(b)}{1-U_{x,g}(b)}$ so that

$$\begin{aligned}(U(x), 0, 1) &= (U(x), U_{b,g}(b), U_{b,g}(g)) \\ &= c(U_{x,g}(x), U_{x,g}(b), U_{x,g}(g)) + d \\ &= c(0, U_{x,g}(b), 1) + d\end{aligned}$$

for $c = 1 - U_{x,g}(b)$ and $d = U_{x,g}(b)$ (observe that $0 < c, d < 1$).

Thus, for every x, $U(x)$ is the unique number such that the vector $(0, 1, U(x))$ (which is not necessarily an increasing list of numbers) is an increasing affine transformation of $\left(U_{[\{b,g,x\}]}(b), U_{[\{b,g,x\}]}(g), U_{[\{b,g,x\}]}(x)\right)$. Put differently, for each x there exists a unique function $V_{[\{b,g,x\}]} : [\{b,g,x\}] \to \mathbb{R}$ such that (i) $V_{[\{b,g,x\}]}$ is an increasing affine transformation of $U_{[\{b,g,x\}]}$, so that $V_{[\{b,g,x\}]}$ is affine and represents $\succsim$ on $[\{b,g,x\}]$; (ii) $V_{[\{b,g,x\}]}(b) = 0$ and $V_{[\{b,g,x\}]}(g) = 1$. And then $U(x) = V_{[\{b,g,x\}]}(x)$.

We wish to show that U so defined satisfies the two conditions, namely, that it represents preferences and that it is affine. Let there be given $x, y \in X$ and consider the set $Y = \{b, g, x, y\}$. We know that there exists an affine $U_{[Y]}$ that represents preferences on all of $[Y]$, $Y = \{b, g, x, y\}$ included. It has a unique increasing affine transformation, $V_{[Y]}$ that also satisfies $V_{[Y]}(b) = 0$ and $V_{[Y]}(g) = 1$. Consider $z \in [Y]$. We wish to show that $U(z) = V_{[Y]}(z)$. Indeed, we know that $U(z) = V_{[\{b,g,z\}]}(z)$; moreover, $V_{[Y]}$ (weakly) extends $V_{[\{b,g,z\}]}$ from $[\{b,g,z\}]$ to all of $[Y]$; since they are both affine, and both represent preferences on $[\{b,g,z\}]$, with $V_{[\{b,g,x\}]}(b) = V_{[Y]}(b) = 0$ and $V_{[\{b,g,x\}]}(g) = V_{[Y]}(g) = 1$, $V_{[\{b,g,x\}]}(\cdot) = V_{[Y]}(\cdot)$ on $[\{b,g,z\}]$. Hence $V_{[\{b,g,x\}]}(z) = V_{[Y]}(z)$ and $U(z) = V_{[Y]}(z)$ follows. Because $V_{[Y]}$ represents preference on $[Y]$, we have, in particular,

$$x \succsim y \quad \Leftrightarrow \quad V_{[Y]}(x) \geq V_{[Y]}(y) \Leftrightarrow \quad U(x) \geq U(y)$$

hence U represents $\succsim$. To see that U is affine, note that, for $z = \alpha x + (1-\alpha) y$ we have, by Lemmas 17 and 18, $z \in [\{x, y\}] \subset [Y]$. Hence $V_{[Y]}$ is defined on z and it is known to satisfy

$$V_{[Y]}(\alpha x + (1-\alpha) y) = \alpha V_{[Y]}(x) + (1-\alpha) V_{[Y]}(y)$$

hence U also satisfied affinity.

This concludes the proof of existence of U. To see that it is unique up to a positive affine transformation, one need only repeat the argument above: for any affine V that represents $\succsim$ one may apply an affine transformation such that $V(b) = 0$ and $V(g) = 1$, and prove that $V(x) = U(x)$ for all x. $\square$

6 vNM Expected Utility

6.1 Model and Theorem

Let X be a set of alternatives.

The objects of choice are lotteries with finite support. Formally, define

$$L = \left\{ P : X \to [0,1] \;\middle|\; \begin{array}{c} \#\{x|P(x) > 0\} < \infty, \\ \sum_{x \in X} P(x) = 1 \end{array} \right\}.$$

Observe that the expression $\sum_{x \in X} P(x) = 1$ is well-defined thanks to the finite support condition that precedes it.

A mixing operation is performed on L, defined for every $P, Q \in L$ and every $\alpha \in [0,1]$ as follows: $\alpha P + (1-\alpha)Q \in L$ is given by

$$(\alpha P + (1-\alpha)Q)(x) = \alpha P(x) + (1-\alpha)Q(x)$$

for every $x \in X$. The intuition behind this operation is of conditional probabilities: assume that I offer you a compound lottery that will give you the lottery P with probability α, and the lottery Q – with probability $(1-\alpha)$. Asking, what is the probability to obtain a certain outcome x, one observes that it is, indeed, α times the conditional probability of x if one gets P plus $(1-\alpha)$ times the conditional probability of x is one gets Q.

Since the objects of choice are lotteries, the observable choices are modeled by a binary relation on L, $\succsim \subset L \times L$. The vNM axioms are:

V1. **Weak order**: $\succsim$ is complete and transitive.

V2. **Continuity**: For every $P, Q, R \in L$, if $P \succ Q \succ R$, there exist $\alpha, \beta \in (0,1)$ such that

$$\alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R.$$

V3. **Independence**: For every $P, Q, R \in L$, and every $\alpha \in (0,1)$,

$$P \succ Q \qquad \textit{implies} \qquad \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R.$$

Theorem 22 *(vNM)* $\succsim \subset L \times L$ *satisfies V1-V3 if and only if there exists* $u : X \to \mathbb{R}$ *such that, for every* $P, Q \in L$

$$P \succsim Q \quad \textit{iff} \quad \sum_{x \in X} P(x)u(x) \geq \sum_{x \in X} Q(x)u(x).$$

Moreover, in this case u *is unique up to a positive affine transformation (pat).*

6.2 Proof

This is clearly an example of Theorem 16, which was actually a generalization of the present one. By Theorem 16, we have a representation by an affine U on L. It remains to define, for $x \in X$,

$$u(x) = U([x])$$

where $[x] \in L$ is the lottery that assigns probability 1 to the outcome x. By inductive application of affinity we find that, for any lottery P,

$$U(P) = \sum_{\{x|P(x)>0\}} P(x)\,U([x]) = \sum_{x \in X} P(x)u(x).$$

To see that u is unique up to positive affine transformations, observe that an affine transformation of U defines an affine transformation of u and vice versa. □

The first lemmas of Theorem 16 are needed whichever way we look at the vNM or Herstein-Milnor theorems. However, once we established these, when the time comes to define the utility function, there are two other ways to continue. The proof provided above is relatively general, yet it makes use of very little machinery. Moreover, it has the advantage of mimicking a process by which the decision maker's utility is calibrated. However, this proof does not shed much light on the geometry of preferences. The following approaches add something in this respect.

6.3 A geometric approach

To understand the geometry of the independence axiom, it is useful to consider the case in which X contains only three pairwise-non-equivalent outcomes. Say,

$X = \{x_1, x_2, x_3\}$ where $x_1 \succ x_2 \succ x_3$. Every lottery in L is a vector (p_1, p_2, p_3) such that $p_i \geq 0$ and $p_1 + p_2 + p_3 = 1$. For visualization, let us focus on the probabilities of the best and worst outcomes. Formally, consider the $p_1 p_3$ plane: draw a graph in which the x axis corresponds to p_1 and the y axis – to p_3. The *Marschak-Machina Triangle* is

$$\Delta = \{(p_1, p_3) \,|\, p_1, p_3 \geq 0,\ p_1 + p_3 \leq 1\}.$$

Thus, the point $(1, 0)$ corresponds to the best lottery x_1 (with probability 1), $(0, 0)$ – to x_2, and $(0, 1)$ – to the worst lottery x_3. Every lottery P corresponds to a unique point (p_1, p_3) in the triangle, and vice versa. We will refer to the point (p_1, p_3) by P as well.

Consider the point $(0, 0)$. By reasoning as in the previous proof, we conclude that, along the segment connecting $(1, 0)$ with $(0, 1)$ there exists a unique point which is equivalent to $(0, 0)$. Such a unique point will exist along the segment connecting $(1, 0)$ with $(0, c)$ for every $c \in [0, 1]$. The continuity axiom implies (in the presence of the independence axiom) that these points generate a continuous curve, which is the indifference curve of x_2.

Lemmas 17 and 18 imply that the indifference curves are linear. (Otherwise, they will have to be "thick", and for some c we will obtain intervals of indifference on the segment connecting $(1, 0)$ with $(0, c)$.) We want to show that they are also parallel.[3]

Consider two lotteries $P \sim Q$. Consider another lottery R such that $S = R + (Q - P)$ is also in the triangle. (In this equation, the points are considered as vectors in Δ.) We claim that $R \sim S$. Indeed, if, say $R \succ S$ the independence axiom would have implied $\frac{1}{2}R + \frac{1}{2}Q \succ \frac{1}{2}S + \frac{1}{2}Q$, and, by $P \sim Q$, also $\frac{1}{2}S + \frac{1}{2}Q \sim \frac{1}{2}S + \frac{1}{2}P$. We would have obtained $\frac{1}{2}R + \frac{1}{2}Q \succ \frac{1}{2}S + \frac{1}{2}P$ while we know that these two lotteries are identical. (Not only equivalent, simply equal, because $S + P = R + Q$.) Similarly $S \succ R$ is impossible. That is, the line segment

[3] You may suggest that linear indifference curves that are not parallel would intersect, contradicting transitivity. But if the intersection is outside the triangle, such preferences may well be transitive. See Chew (1983) and Dekel (1986).

connecting R and S is also an indifference curve. However, by $P - Q = R - S$ we realize that the indifference curve going through R, S is parallel to the one going through P, Q. This argument can be repeated for practically every R if Q is sufficiently close to P. (Some care is needed near the boundaries.) Thus all indifference curves are linear and parallel.

The Independence axiom might bring to mind some high school geometry. Geometrically, the Independence axiom states that indifference curves should be parallel: consider P, Q, R, and draw a triangle whose base is PQ and whose apex is R. Assume that $P\tilde{}Q$ so that the base of the triangle is an indifference curve. Then, when you consider points on the edges PR and QR that are proportionately removed from $P(Q)$ in the direction of R – that is, $\alpha P + (1-\alpha)R$ and $\alpha Q + (1-\alpha)R$ – you find that, by the Independence axiom, they are also equivalent to each other. Thus, the segment connecting them is also part of an indifference curve. But the proportionality means that we generated similar triangles, and their bases are parallel.

Once we know that the indifference curves are linear and parallel, we're more or less done: linear and parallel lines can be described by a single linear function. That is, one can choose two numbers a_1 and a_3 such that all the indifference curves are of the form $a_1 p_1 + a_3 p_3 = c$ (varying the constant c from one curve to the other). Setting $u(x_1) = a_1$, $u(x_2) = 0$, and $u(x_3) = a_3$, this is an expected utility representation.

This argument can be repeated for any finite set of outcomes X. "Patching" together the representations for all the finite subsets is done in the same way as in the algebraic approach.

6.4 A separation argument

It is worth noticing that the vNM theorem is basically a separating hyperplane theorem. To see the gist of the argument, assume that X is finite, though the same idea applies more generally. Embed L in $\mathbb{R}^X$, so that we have a linear space, and we can discuss, for $P, Q \in L \subset \mathbb{R}^X$ also the difference $P - Q \in \mathbb{R}^X$.

Consider the sets

$$A = \left\{ P - Q \in \mathbb{R}^X \,|P \succsim Q \right\}$$

and

$$B = \left\{ P - Q \in \mathbb{R}^X \,|Q \succ P \right\}.$$

We first show that $R \succsim S$ if and only if $(R - S) \in A$. This is true because, if $R - S = P - Q$, we find, by reasoning similar to that used above, that $P \succsim Q$ iff $R \succsim S$. Similarly, $S \succ R$ if and only if $(R - S) \in B$.

Next we show that both A and B are convex. This is again an implication of the independence axiom: suppose, say, $(P - Q), (R - S) \in A$ and consider

$$\alpha(P - Q) + (1 - \alpha)(R - S) = (\alpha P + (1 - \alpha)R) - (\alpha Q + (1 - \alpha)S).$$

$P \succsim Q$ and $R \succsim S$ imply (by two applications of the independence axiom)

$$\alpha P + (1 - \alpha)R \succsim \alpha Q + (1 - \alpha)R \succsim \alpha Q + (1 - \alpha)S$$

which means that $(\alpha P + (1 - \alpha)R) - (\alpha Q + (1 - \alpha)S) \in A$. The same reasoning applies to B.

Finally, we need to show that A is closed and that B is open. The topology in which such claims would be true is precisely the topology in which the continuity axiom guarantees continuity of preferences: an open neighborhood of a point P is defined by

$$\cup_{R \in L} \left\{ \alpha P + (1 - \alpha)R \,|\, \alpha \in [0, \varepsilon_R) \right\}$$

where, for every $R \in L$, $\varepsilon_R > 0$. You may verify that this topology renders vector operations continuous. (Observe that this is not the standard topology on $\mathbb{R}^X$, even if X is finite, because ε_R need not be bounded away from 0. That is, as we change the "target" R, the length of the interval coming out of P in the direction of R, still inside the neighborhood, changes and may converge to zero. Still, in each given direction $R - P$ there is an open segment, leading from P towards R, which is in the neighborhood.)

When we separate A from B by a linear functional, we can refer to the functional as the utility function u. Linearity of the utility with respect to the probability values guarantees affinity, i.e., that

$$u(\alpha P + (1-\alpha)R) = \alpha u(P) + (1-\alpha)u(R).$$

Since every P has a finite support, using this property inductively results in the expected utility formula.

7 de Finetti's Theorem

7.1 Model and Theorem

Let the set of outcomes be $\mathbb{R}$, interpreted as monetary values. We are after an expected-value-maximization theorem, which corresponds to a decision maker who is risk neutral. While this is not very realistic, the theorem is important on the way to the following results. One can also try to re-interpret the numerical values as "utiles", so that the result is expected utility, rather than expected value maximization.

The objects of choice are *acts*, which are functions from states of the world to the outcomes $\mathbb{R}$. Importantly, there are no probabilities given. The probability distribution over the states of the world will be derived from preferences, and will be interpreted as subjective probabilities of the decision maker.

Let S be a finite set of states, $S = \{1, 2, ..., n\}$, where the alternatives are all the real-valued functions on S: $X = \mathbb{R}^S$. Assume that the decision maker has a preference order over bets, $\succsim \subset X \times X$. Consider the following axioms:

D1. **Weak order**: $\succsim$ is complete and transitive.

D2. **Continuity**: For every $x \in X$, the sets $\{\, y \,|\, x \succ y\}$, $\{\, y \,|\, y \succ x\}$ are open[4].

D3. **Additivity**: For every $x, y, z \in X$, $x \succsim y$ iff $x + z \succsim y + z$.

D4. **Monotonicity**: For every $x, y \in X$, $x_i \geq y_i$ for all $i \leq n$ implies $x \succsim y$.

D5. **Non-triviality**: There exist $x, y \in X$ such that $x \succ y$.

Theorem 23 *(de Finetti)* $\succsim \subset X \times X$ *satisfies D1-D5 if and only if there exists a probability vector* $p \in \Delta^{n-1}$ *such that, for every* $x, y \in X$,

$$x \succsim y \quad \textit{iff} \quad px \geq py.$$

Moreover, in this case p is unique.

[4] Here we refer to the standard topology on $\mathbb{R}^n$. The condition is therefore identical to the continuity of consumer preferences in Debreu (1959).

As a reminder, Δ^{n-1} is the set of probability vectors on $\{1, ..., n\}$. The notation px refers to the inner product, that is, $\sum_i p_i x_i$, which is the expected payoff of x relative to the probability p.

7.2 Proof

Let us first show that D1-D3 are equivalent to the existence of $p \in \mathbb{R}^n$ such that

$$x \succsim y \quad \text{iff} \quad px \geq py$$

for every $x, y \in X$.

Necessity of the axioms is immediate. To prove sufficiency, observe first that, for every $x, y \in X$,

$$x \succsim y \quad \text{iff} \quad x - y \succsim 0.$$

Define

$$A = \{x \in X \, | x \succsim 0\}$$

and

$$B = \{x \in X \, | 0 \succ x\}.$$

Clearly, $A \cap B = \varnothing$ and $A \cup B = X$. Also, A is closed and B is open. If $B = \varnothing$, $p = 0$ is the vector we need. Otherwise, both A and B are non-empty.

We wish to show that they are convex. To this end, we start by observing that, if $x \succsim y$, then $x \succsim z \succsim y$ where $z = \frac{x+y}{2}$. This is true because, defining $d = \frac{y-x}{2}$, we have $x+d = z$ and $z+d = y$. D3 implies that $x \succsim z \Leftrightarrow x+d \succsim z+d$, i.e. $x \succsim z \Leftrightarrow z \succsim y$. Hence $z \succ x$ would imply $y \succ z$ and $y \succ x$, a contradiction. Hence $x \succsim z$, and $z \succsim y$ follows from $x \succsim z$.

Next we wish to show that if $x \succsim y$, then $x \succsim z \succsim y$ for any $z = \lambda x + (1-\lambda)y$ with $\lambda \in [0,1]$. If λ is a binary rational (i.e., of the form $k/2^i$ for some $k, i \geq 1$), the conclusion follows from an inductive application of the previous claim (for $\lambda = 1/2$). As for other values of λ, $z \succ x$ ($y \succ z$) would imply, by continuity, the same preference in an open neighborhood of z, including binary rationals.

It follows that one can separate A from B by a linear function. That is, there exists a linear $f : X \to \mathbb{R}$ and a number $c \in \mathbb{R}$ such that

$$x \in A \quad \text{iff} \quad f(x) \geq c$$

(and $x \in B$ iff $f(x) < c$). Since $0 \in A$, $c \leq 0$. If $c < 0$, consider x with $f(x) = \frac{3c}{4}$. Then $x \in A$ but $2x \in B$. That is, $x \succsim 0$ but $0 \succ 2x$, in contradiction to D3 (coupled with transitivity). This implies that $c = 0$. Denoting $p_i = f(e_i)$ (where e_i is the i-th unit vector), we obtain

$$\begin{array}{rrcl} & x & \succsim & y \\ \text{iff} & x - y & \succsim & 0 \\ \text{iff} & x - y & \in & A \\ \text{iff} & f(x-y) & \geq & 0 \\ \text{iff} & px & \geq & py. \end{array}$$

It is easily verifiable that, given the above, D4 is equivalent to $p_i \geq 0$, and D5 – to the claim that $p \neq 0$, or $\sum_i p_i > 0$. Under this conditions, p can be normalized to be a probability vector, and it is the unique probability vector representing preference as above. □

8 Anscombe-Aumann's Theorem

8.1 Model and Theorem

Anscombe-Aumann's model has states of the world, and derives subjective probabilities on them, as does de Finetti's. However, in this model it is not assumes that the outcomes are real numbers; rather, the outcomes are vNM lotteries. So we have two levels of uncertainty: first, we do not know which state of the world will obtain, and we don't even have a probability for that uncertainty. Second, given a state, the decision maker will be facing a lottery with known, objective probabilities as in the vNM model.

Formally, we use we the set-up introduced by Fishburn (1970). As a reminder, the vNM lotteries are

$$L = \left\{ P : X \to [0,1] \;\middle|\; \begin{array}{c} \#\{x|P(x) > 0\} < \infty, \\ \sum_{x \in X} P(x) = 1 \end{array} \right\}$$

and this set is endowed with a mixing operation: for every $P, Q \in L$ and every $\alpha \in [0,1]$, $\alpha P + (1-\alpha)Q \in L$ is given by

$$(\alpha P + (1-\alpha)Q)(x) = \alpha P(x) + (1-\alpha)Q(x).$$

The state space is S. We wish to state that acts are functions from S to L. In general we would need to endow S with a σ-algebra, and deal with measurable and bounded acts. Both of these terms have to be defined in terms of preferences, because we don't have yet a utility function. Instead, we will simplify our lives and assume that S is finite. However, the theorem holds also for general measurable spaces.

The set of acts is $F = L^S$. We will endow F with a mixture operation as well, performed pointwise. That is, for every $f, g \in F$ and every $\alpha \in [0,1]$, $\alpha f + (1-\alpha)g \in F$ is given by

$$(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \qquad \forall s \in S.$$

We will denote the decision maker's preference order by $\succsim \subset F \times F$ and we will abuse this notation as usual. In particular, we can write, for $P, Q \in L$,

$P \succsim Q$, understood as $f_P \succsim f_Q$ where, for every $R \in L$, $f_R \in F$ is the constant act given by $f_R(s) = R$ for all $s \in S$.

The interpretation is that, if the decision maker chooses $f \in F$ and Nature chooses $s \in S$, a roulette wheel is spun, with distribution $f(s)$ over the outcomes X, so that your probability to get outcome x is $f(s)(x)$.

For a function $u : X \to \mathbb{R}$ we will use the notation

$$E_P u = \sum_{x \in X} P(x)u(x)$$

for $P \in L$.

Thus, if you choose $f \in F$ and Nature chooses $s \in S$, you will get a lottery $f(s)$, which has the expected u-value of

$$E_{f(s)} u = \sum_{x \in X} f(s)(x)u(x).$$

Anscombe-Aumann's axioms are the following. The first three are identical to the vNM axioms. Observe that they now apply to more complicated creatures: rather than to specific vNM lotteries, we now deal with functions whose values are such lotteries, or, if you will, with vectors of vNM lotteries, indexed by the state space S. The next two axioms are almost identical to de Finetti's last two axioms, guaranteeing monotonicity and non-triviality:

AA1. **Weak order**: $\succsim$ is complete and transitive.

AA2. **Continuity**: For every $f, g, h \in F$, if $f \succ g \succ h$, there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h.$$

AA3. **Independence**: For every $f, g, h \in F$, and every $\alpha \in (0, 1)$,

$$f \succ g \qquad \textit{implies} \qquad \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h.$$

AA4. **Monotonicity**: For every $f, g \in F$, $f(s) \succsim g(s)$ for all $s \in S$ implies $f \succsim g$.

AA5. **Non-triviality**: There exist $f, g \in X$ such that $f \succ g$.

Theorem 24 *(Anscombe-Aumann)* $\succsim$ *satisfies AA1-AA5 if and only if there exist a probability measure* μ *on* S *and a non-constant function* $u : X \to \mathbb{R}$ *such that, for every* $f, g \in F$

$$f \succsim g \quad \textit{iff} \quad \int_S (E_{f(s)} u) d\mu(s) \geq \int_S (E_{g(s)} u) d\mu(s)$$

Furthermore, in this case μ *is unique, and* u *is unique up to positive linear transformations.*

8.2 Proof

The first part of the proof is a direct application of Theorem 16. The objects of choice can be thought of as matrices whose columns are states in S and their rows are outcomes in X. For the sake of the concreteness, let's assume that X is also finite, as is S. Then, every act f can be thought of as a matrix of non-negative numbers, such that in each column (that is, for every state s), the numbers sum up to 1 (defining a probability distribution over the outcomes in X). Viewed thus, an act f is an extreme point of the set F if, at each and every column s, it assigns probability 1 to an outcome x. Thus, there are $|X|^{|S|}$ extreme points, and F is their convex hull.

The first three axioms mean that we can have a representation of $\succsim$ by an affine function U. We now wish to show that this affine function can be represented as

$$U(f) = \sum_{x,s} f(s)(x)\, u(x,s)$$

for some $u : X \times S \to \mathbb{R}$.

Lemma 25 *There exists* $u : X \times S \to \mathbb{R}$ *such that*

$$U(f) = \sum_{x,s} f(s)(x)\, u(x,s) \qquad \forall f \in F \tag{7}$$

Proof: Observe that, if the domain of U were all the functions $f : X \times S \to \mathbb{R}_+$ such that $\sum_{x,s} f(s)(x) = |S|$, (or, equivalently, all the distributions on the matrix, that is, all $f : X \times S \to \mathbb{R}_+$ such that $\sum_{x,s} f(s)(x) = 1$) then each extreme point would be assigning positive weight to one pair (x, s) only, and the proof would be immediate. However, the set F has the additional constraint that $\sum_x f(s)(x) = 1$ for each s separately, and this means that it has many more extreme points and a bit more needs to be said to obtain (7).

Let us choose $x_* \in X$ and shift U so that $U([x_*], ..., [x_*]) = 0$. This can be done without loss of generality. Next, define, for each s,

$$u(x_*, s) = 0.$$

We have to define $u(x, s)$ for other x's. It will be convenient to define

$$u_s(\cdot) \equiv u(\cdot, s) : L \to \mathbb{R}$$

that is, to have u_s be defined for all lotteries on X, with $u(x, s) = u([x], s) = u_s([x])$, that is, to define u_s in such a way that the degenerate lottery $[x]$, assigning probability 1 to x, has the same value as the outcome x. (Obviously, this is an abuse of notation, but we're accustomed to such sins by now.)

For $P \in L$, $s \in S$, define

$$h_{P,s}(s') = \begin{cases} P & s' = s \\ [x_*] & s' \neq s \end{cases}$$

and

$$u_s(P) = U(h_{P,s}).$$

That is, $u_s(P)$ is the U value of the act f that obtains $[x_*]$ at each state $s' \neq s$ and takes the value P at s:

$$u_s(P) = U([x_*], ..., [x_*], P, [x_*], ..., [x_*]).$$

We argue that, for every f,

$$U(f) = \sum_s u_s(f(s)).$$

To see this, let there be given $f \in F$. Consider the act $f' \in F$ defined by

$$f'(s) = \frac{1}{n} f(s) + \left(1 - \frac{1}{n}\right)[x_*]$$

where $n = |S|$.

We can think of f' as the mixture of f and $[x_*]$: clearly, because of our definition of the mixture operation as a pointwise operation,

$$f' = \frac{1}{n} f + \left(1 - \frac{1}{n}\right) \left([x_*], ..., [x_*]\right)$$

This means, by affinity of U, that

$$U\left(f'\right) = \frac{1}{n} U\left(f\right) + \left(1 - \frac{1}{n}\right) U\left([x_*], ..., [x_*]\right)$$

recalling that $U\left([x_*], ..., [x_*]\right) = 0$ as have

$$U\left(f'\right) = \frac{1}{n} U\left(f\right). \tag{8}$$

On the other hand, we can also think of f' as the n-fold mixture of acts, each of which equals $[x_*]$ in all but one state. Formally, define $g_s \in F$ by

$$g_s\left(s'\right) = h_{f(s),s} = \begin{cases} f\left(s\right) & s' = s \\ [x_*] & s' \neq s \end{cases}$$

and observe that

$$f' = \sum \frac{1}{n} g_s$$

which implies

$$U\left(f'\right) = \sum \frac{1}{n} U\left(g_s\right). \tag{9}$$

Comparing (8) and (9) we get

$$\frac{1}{n} U\left(f\right) = \sum \frac{1}{n} U\left(g_s\right)$$

and

$$U\left(f\right) = \sum U\left(g_s\right).$$

Next, note that, by definition of g_s, which is $\left([x_*], ..., [x_*], f(s), [x_*], ..., [x_*]\right)$, and the definition of $u_s\left(\cdot\right)$, we get

$$U\left(g_s\right) = u_s\left(f\left(s\right)\right)$$

so that

$$U\left(f\right) = \sum_s u_s\left(f\left(s\right)\right).$$

It only remains to note that, at each and every state s,

$$u_s(f(s)) = \sum_x f(s)(x)\, u(x,s)$$

as in the reasoning in the vNM case (where affinity of U yields the results immediately as the extreme points are the degenerate lotteries). $\square$

It follows that, for $x \in X$ and $s \in S$ there is a number $u(x,s)$ such that

$$f \succsim g \qquad iff \qquad \sum_{x,s} f(s)(x)\, u(x,s) \geq \sum_{x,s} g(s)(x)\, u(x,s) \qquad \forall f,g \in F. \tag{10}$$

Observe that, for every vector of real numbers $(\beta_s)_{s \in S}$, if we define

$$v(x,s) = u(x,s) + \beta_s$$

then we get a matrix v that also satisfies (10): indeed, for every $f \in F$,

$$\begin{aligned}
\sum_{x,s} f(s)(x)\, v(x,s) &= \sum_{x,s} f(s)(x)\left[u(x,s) + \beta_s\right] \\
&= \sum_{x,s} f(s)(x)\, u(x,s) + \sum_{x,s} f(s)(x)\, \beta_s \\
&= \sum_{x,s} f(s)(x)\, u(x,s) + \sum_s \beta_s \sum_x f(s)(x) \\
&= \sum_{x,s} f(s)(x)\, u(x,s) + \sum_s \beta_s
\end{aligned}$$

because, for every $f \in F$ and $s \in S$, $f(s)$ is a vNM lottery, so that $\sum_x f(s)(x) = 0$. Thus, shifting the utility numbers $u(x,s)$ by a constant β_s in column s (for every x) results in a shift of $U(f) = \sum_{x,s} f(s)(x)\, u(x,s)$ and thus in a new matrix that still represents preferences as in (10).

Let us pick an outcome $x_* \in X$ and henceforth assume that $u(x_*, s) = 0$ for all s. In view of the above, this restriction entails no loss of generality. One may verify that the remaining degree of freedom is only a positive multiplication of all $\{u(x,s)\}_{x,s}$ (by the same positive number).

The idea of the proof is to show that the functions on X defined by

$$u_s(x) = u(x,s)$$

are non-negative multiples of a single function $u : X \to \mathbb{R}$. More precisely, we will distinguish between two type of states: those that are "null", intuitively corresponding to having a zero subjective probability, and that do not matter for the decision, and those that are "non-null", intuitively corresponding to positive subjective probabilities. For any two non-null states,s, s', we wish to show that $u_{s'}$ is a positive multiple of u_s. Then, we can fix one function $u : X \to \mathbb{R}$ and write $u(x, s) = u_s(x) = \mu_s u(x)$ for some $\mu_s > 0$. Without loss of generality, assume that we normalized the coefficients μ_s so that they sum up to 1. This allows us to think of them as probabilities, writing

$$\begin{aligned} \sum_{x,s} f(s)(x)\, u(x,s) &= \sum_{x,s} f(s)(x)\, \mu_s u(x) \\ &= \sum_{s} \mu_s \sum_{x,s} f(s)(x)\, u(x) \\ &= \sum_{s} \mu_s \left(E_{f(s)} u\right) \end{aligned}$$

namely, the expected utility of u, where the inner expression is the expectation relative to the objective probabilities given by the lottery $f(s)$, and all these are integrated over with respect to the probability vector μ, interpreted as the decision maker's subjective probability over the state space S.

Formally, for $s \in S$, define a function $u_s : X \to \mathbb{R}$ by

$$u_s(x) = u(x, s)$$

and define also $u : X \to \mathbb{R}$ by

$$u(x) = \sum_{s \in S} u_s(x).$$

Observe that, in the definition of monotonicity, the relation $f(s) \succsim g(s)$ means that the (constant) act, which is equal to $f(s)$ at each $s' \in S$, is at least as good as the constant act corresponding to $g(s)$. Using the representation (10), this is true iff

$$\sum_{x,s} f(s)(x)\, u(x, s') \geq \sum_{x,s} g(s)(x)\, u(x, s').$$

Because $f(s)(x)$ and $g(s)(x)$ are independent of s, this can be written as

$$\sum_x f(s)(x) \sum_s u(x,s) \geq \sum_x g(s)(x) \sum_s u(x,s)$$

that is,

$$\sum_x f(s)(x)\, u(x) \geq \sum_x g(s)(x)\, u(x).$$

In other words, the sum of state-utilities, $u = \sum_{s\in S} u_s$ is a vNM function that represents preferences over constant acts. We now wish to show that for every s there is $\mu_s \geq 0$ such that $u_s(\cdot) = \mu_s u(\cdot)$.

What will use the following lemma.

Lemma 26 *Let $a, b \in \mathbb{R}^n$ be such that*

$$az \geq 0 \Rightarrow bz \geq 0$$

for every $z \in \mathbb{R}^n$ with $\sum_i z_i = 0$. Then there are $\lambda \geq 0$ and $c \in \mathbb{R}$ such that

$$b_i = \lambda a_i + c$$

for every $i \leq n$.

Proof: We use the duality theorem of linear programming. Consider the problem

$$Min_{z=(z_1,\ldots,z_n)} bz \qquad \text{((P))}$$

subject to

$$\begin{aligned} az &\geq 0 \\ 1z &= 0 \end{aligned}$$

where 1 stands for the vector of 1's.

Observe that, since the feasible set is homogeneous (z is feasible iff αz if feasible for all $\alpha > 0$), (P) is bounded (from below) iff it is bounded (from below) by zero. Indeed, what we are told is precisely that it is bounded by zero: $az \geq 0$ and $1z = 0$ imply $bz \geq 0$. Hence

$$az \geq 0 \Rightarrow bz \geq 0 \;\forall z \in \mathbb{R}^n, 1z = 0$$

is equivalent to (P) being bounded, which is equivalent to its dual being feasible. The dual will have two variables – say, λ for the first constraint and c for the second. Its objective function is

$$0\lambda + 0c = 0$$

and it will be a maximization problem. Further, λ will be non-negative (because it is attached to a $\geq$ constraint in (P)) and c will be unconstrained, because it is attached to an equality constraint. Finally, because the variables z_i are not constrained to be positive or negative, the constraints in the dual problem will be equality constraints. To sum, the dual is

$$\begin{aligned} & Maz_{\lambda,c} 0 \\ & \text{subject to} \\ \lambda a_i + c &= b_i \quad \forall i \end{aligned}$$

Finally, this problem is feasible iff there are $\lambda \geq 0$ and $c \in \mathbb{R}$ such that

$$b_i = \lambda a_i + c$$

which is what we set out to prove. $\square$

Let $X = \{x_1, ..., x_n\}$. Let $a_i = u(x_i)$ for $i \leq n$. For $s \in S$, define $b_i = u_s(x_i)$ for $i \leq n$. Consider two acts $f, g \in F$ such that $f(s') = g(s')$ for $s' \neq s$. Observe that $f(s) \succsim g(s)$ iff

$$\sum_x f(s)(x) u(x) \geq \sum_x g(s)(x) u(x)$$

or

$$\sum_x [f(s)(x) - g(s)(x)] u(x) \geq 0$$

that is,

$$[f(s)(\cdot) - g(s)(\cdot)] a \geq 0.$$

Monotonicity implies, that whenever this is the case ($f(s) \succsim g(s)$), we have $f \succsim g$. However, $f \succsim g$ is equivalent to

$$\sum_{x,s'} f(s')(x) u(x,s') \geq \sum_{x,s'} g(s')(x) u(x,s')$$

and, since $f(s') = g(s')$ for $s' \neq s$, also to

$$\sum_x f(s)(x) u_s(x) \geq \sum_x g(s)(x) u_s(x)$$

or

$$\sum_x [f(s)(x) - g(s)(x)] u_s(x) \geq 0$$

that is,

$$[f(s)(\cdot) - g(s)(\cdot)] b \geq 0.$$

Consider a vNM lottery P such that $P(x) = 1/n$ for all x. Select f such that $f(s) = P$. For $z \in \left(-\frac{1}{n}, \frac{1}{n}\right)^n$, $1z = 0$, select g such that $f(s') = g(s')$ for $s' \neq s$ and $g(s)(x) = \frac{1}{n} - z$. So that $f(s)(\cdot) - g(s)(\cdot) = z$. For every such z we therefor get that $az \geq 0 \Rightarrow bz \geq 0$. Due to homogeneity, this also implies that $az \geq 0 \Rightarrow bz \geq 0$ holds for every vector $z \in \mathbb{R}^n, 1z = 0$. By the lemma, we have $\lambda \geq 0$ and $c \in \mathbb{R}$ such that

$$u_s(x_i) = \lambda u(x_i) + c.$$

Plugging in x_* (which equals one of the x_i) we obtain $c = 0$. Thus, for every s there exists $\lambda_s \geq 0$ such that

$$u_s(x) = \lambda_s u(x) \qquad \forall x \in X.$$

It remains to normalize the coefficients $(\lambda_s)_s$ to obtain a probability vector $(\mu_s)_s$. To see that this can be done, we need to guarantee that not all of them are zero.

Define a state $s \in S$ to be *null* if, whenever $f(s') = g(s')$ for all $s' \neq s$, $f \sim g$. Observe that if all states were null, then, by replacing $f(s)$ by $g(s)$ consecutively, we can prove that $f \sim g$ for all $f, g \in F$, contradiction the non-triviality axiom A5. Hence, not all states are null.

It is easy to see that, if s is null, then $u_s(\cdot) = u(\cdot, s)$ has to be a constant on X. Since we have $u(x_*, s) = 0$, this constant is zero, that is, $u_s(\cdot) = u(\cdot, s) = 0$ for all null states s. Thus, if s is null, it has to be the case that $\lambda_s = 0$.

Conversely, if $\lambda_s = 0$ it follows that $u_s(x)$ vanishes for all x, and then s is null. Thus, $\lambda_s > 0$ iff s is non-null (and $\lambda_s = 0$ iff s is null). Since there are non-null states, $\sum_s \lambda_s > 0$ and

$$\mu_s = \frac{\lambda_s}{\sum_s \lambda_s}$$

defines a probability vector such that

$$\sum_{x,s} f(s)(x)\, u(x,s) = \sum_s \mu_s \left(E_{f(s)} u\right)$$

for all $f \in F$.

9 Savage's Theorem

9.1 Set-up

Savage's model includes two primitive concepts: states and outcomes. The set of *states*, S, should be thought of as an exhaustive list of all scenarios that might unfold. An *event* is any subset $A \subset S$. There are no measurability constraints, and S is not endowed with an algebra of measurable events. If you wish to be more formal about it, you can define the set of events to be the maximal σ-algebra, $\Sigma = 2^S$, with respect to which all subsets are measurable.

The set of *outcomes* will be denoted by X. An outcome x is assumed to specify all that is relevant to your well-being, insomuch as it may be relevant to your decision.

The objects of choice are acts, which are defined as functions from states to outcomes, and denoted by F. That is,

$$F = X^S = \{f \mid f : S \to X\}.$$

Acts whose payoffs do not depend on the state of the world s are constant functions in F. We will abuse notation and denote them by the outcome they result in. Thus, $x \in X$ is also understood as $x \in F$ with $x(s) = x$.

Since the objects of choice are acts, Savage assumes a binary relation $\succsim \subset F \times F$. The relation will have its symmetric and asymmetric parts, $\sim$ and $\succ$, defined as usual. It will also be extended to X with the natural convention. Specifically, for two outcomes $x, y \in X$, we say that $x \succsim y$ if and only if the constant function that yields always x is related by $\succsim$ to the constant function that yields always y.

For two acts $f, g \in F$ and an event $A \subset S$, define an act f_A^g by

$$f_A^g(s) = \begin{cases} g(s) & s \in A \\ f(s) & s \in A^c \end{cases}.$$

Think of f_A^g as "f, where on A we replaced it by g".

An event A is *null* if, for every $f, g \in F$, $f \sim_A g$. That is, if you know that f and g yield the same outcomes if A does not occur, you consider them equivalent.

9.2 Axioms

P1 $\succsim$ is a weak order.

P2 For every $f, g, h, h' \in F$, and every $A \subset S$,

$$f^h_{A^c} \succsim g^h_{A^c} \quad \text{iff} \quad f^{h'}_{A^c} \succsim g^{h'}_{A^c}.$$

P3 For every $f \in F$, non-null event $A \subset S$ and $x, y \in X$,

$$x \succsim y \quad \text{iff} \quad f^x_A \succsim f^y_A.$$

P4 For every $A, B \subset S$ and every $x, y, z, w \in X$ with $x \succ y$ and $z \succ w$,

$$y^x_A \succsim y^x_B \quad \text{iff} \quad w^z_A \succsim w^z_B.$$

P5 There are $f, g \in F$ such that $f \succ g$.

P6 For every $f, g, h \in F$ with $f \succ g$ there exists a partition of S, $\{A_1, ..., A_n\}$ such that, for every $i \leq n$,

$$f^h_{A_i} \succ g \quad \text{and} \quad f \succ g^h_{A_i}.$$

P7 For every $f, g \in F$ and event $A \subset S$, if, for every $s \in A$, $f \succsim_A g(s)$, then $f \succsim_A g$, and if, for every $s \in A$, $g(s) \succsim_A f$, then $g \succsim_A f$.

9.3 Results

9.3.1 Finitely additive measures

Normally, when you study probability, you define a measure μ to be a probability on a measurable space (Ω, Σ) if it is a function $\mu : \Sigma \to \mathbb{R}_+$ such that

$$\begin{aligned} \mu\left(\cup_i^\infty A_i\right) &= \sum_i^\infty \mu\left(A_i\right) \qquad (11) \\ \text{whenever} \quad i &\neq j \Rightarrow A_i \cap A_j = \varnothing \end{aligned}$$

and $\mu(\Omega) = 1$. Condition (11) is referred to as σ-additivity.

Finite additivity is the condition known as $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \varnothing$, which is clearly equivalent to (11) if you replace ∞ by any finite n:

$$\begin{aligned} \mu\left(\cup_i^n A_i\right) &= \sum_i^n \mu\left(A_i\right) \qquad (12) \\ \text{whenever} \quad i &\neq j \Rightarrow A_i \cap A_j = \varnothing \end{aligned}$$

If you already have a finitely additive measure, σ-additivity is an additional constraint of continuity: define $B_n = \cup_i^n A_i$ and $B = \cup_i^\infty A_i$. Then $B_n \nearrow B$ and (11) means

$$\mu\left(\lim_{n\to\infty} B_n\right) = \mu\left(\cup_i^\infty A_i\right) = \sum_i^\infty \mu\left(A_i\right) = \lim_{n\to\infty} \mu\left(B_n\right)$$

that is, σ-additivity of μ is equivalent to saying that the measure of the limit is the limit of the measure, when increasing sequences of events are concerned.

9.3.2 Non-atomic measures

In the standard case of a σ-additive measure: an event A is an atom of μ if

(i) $\mu(A) > 0$

(ii) For every event $B \subset A$, $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

That is, an atom cannot split, in terms of its probability. When you try to split it to B and $A \backslash B$, you find either that all the probability is on B or that all of it is on $A \backslash B$. A measure that has no atoms is called *non-atomic*.

There are two other possible definitions of non-atomicity, trying to capture the same intuition: you may require, for every event A with $\mu(A) > 0$, that there be an event $B \subset A$ such that $\mu(B)$ is not too close to 0 or to $\mu(A)$. For instance, you may require that

$$\frac{1}{3}\mu(A) \leq \mu(B) \leq \frac{2}{3}\mu(A).$$

Finally, you may consider an even more demanding requirement: that for every event A with $\mu(A) > 0$, and for every $r \in [0, 1]$ there be an event $B \subset A$ such that $\mu(B) = r\mu(A)$.

In the case of a σ-additive μ, all three definitions coincide. But this is not true for finite additivity. Moreover, the condition that Savage needs, and the condition that turns out to follow from P6, is the strongest.

Hence, we will define a finitely additive measure μ to be *non-atomic* if for every event A with $\mu(A) > 0$, and for every $r \in [0, 1]$, there is an event $B \subset A$ such that $\mu(B) = r\mu(A)$.

9.3.3 Savage's Theorem(s)

Theorem 27 *(Savage) Assume that X is finite. Then $\succsim$ satisfies P1-P6 if and only if there exist a non-atomic finitely additive probability measure μ on S (=$(S, 2^S)$) and a non-constant function $u : X \to \mathbb{R}$ such that, for every $f, g \in F$*

$$f \succsim g \quad \text{iff} \quad \int_S u(f(s))d\mu(s) \geq \int_S u(g(s))d\mu(s)$$

Furthermore, in this case μ is unique, and u is unique up to positive linear transformations.

Theorem 28 *(Savage) $\succsim$ satisfies P1-P7 if and only if there exist a non-atomic finitely additive probability measure μ on S (=$(S, 2^S)$) and a non-constant bounded function $u : X \to \mathbb{R}$ such that, for every $f, g \in F$*

$$f \succsim g \quad \text{iff} \quad \int_S u(f(s))d\mu(s) \geq \int_S u(g(s))d\mu(s) \tag{13}$$

Furthermore, in this case μ is unique, and u is unique up to positive linear transformations.

Observe that this theorem restricts u to be bounded. (Of course, this was not stated in Theorem 27 because when X is finite, u is bounded.) The boundedness of u follows from P3. Indeed, if u is not bounded one can generate acts whose expected utility is infinite (following the logic of the St. Petersburg Paradox). This, in and of itself, is not an insurmountable difficulty, but P3 will not hold for such acts: you may strictly improve f from, say, x to y on a non-null event A, and yet the resulting act will be equivalent to the first one, both having infinite expected utility. Hence, as stated, P3 implies that u is bounded. An extension of Savage's theorem to unbounded utilities is provided in Wakker (1993a).[5]

A corollary of the theorem is that an event A is null if and only if $\mu(A) = 0$. In Savage's formulation, this fact is stated on par with the integral representation (13).

9.4 The proof and qualitative probabilities

Savage's proof is too long and involved to be covered here. Savage (1954) develops the proof step by step, alongside conceptual discussions of the axioms. Fishburn (1970) provides a more concise proof, which may be a bit laconic, and Kreps (1988, pp. 115-136) provides more details. Here I will only say a few words about the strategy of the proof, and introduce another concept in this context.

Savage first deals with the case $|X| = 2$. That is, there are two outcomes, say, 1 and 0, with $1 \succ 0$. Thus every $f \in F$ is characterized by an event A, that is, $f = 1_A$. Correspondingly, $\succsim \subset F \times F$ can be thought of as a relation $\succsim \subset \Sigma \times \Sigma$ with $\Sigma = 2^S$.

In this set-up P4 has no bite. Let us translate P1-P3 and P5 to the language of events. P1 would mean, again, that $\succsim$ (understood as a relation on events) is a weak order. P2 is equivalent to the condition:

Cancellation: For every $A, B, C \in \Sigma$, if $(A \cup B) \cap C = \varnothing$, then

$$A \succsim B \quad \text{iff} \quad A \cup C \succsim B \cup C$$

Taken together, P1-P5 are equivalent to:

(i) $\succsim$ is a weak order;

(ii) $\succsim$ satisfies cancellation;

(iii) For every A, $A \succsim \varnothing$;

(iv) $S \succ \varnothing$.

[5] In 1954 Savage was apparently unaware that the boundedness of u follows from P3. Fishburn reports that this became obvious during a discussion they had later on.

A binary relation on an algebra of events that satisfies these conditions was defined by de Finetti to be a *qualitative probability.* The idea was that subjective judgments of "at least as likely as" on events that satisfied certain regularities might be representable by a probability measure, that is, that a probability measure μ would satisfy

$$A \succsim B \quad \text{iff} \quad \mu(A) \geq \mu(B). \tag{14}$$

If such a measure existed, and if it were unique, one could use the likelihood comparisons $\succsim$ as a basis for the definition of subjective probability. Observe that such a definition would qualify as a definition by observable data if you are willing to accept judgments such as "I find A at least as likely as B" as valid data.[6]

de Finetti conjectured that every qualitative probability has a (quantitative) probability measure that represents it. It turns out that this is true if $|S| \leq 4$, but a counterexample can be constructed for $n = 5$. Such a counterexample was found by Kraft, Pratt, and Seidenberg (1959), who also provided a necessary and sufficient condition for the existence of a representing measure.

You can easily convince yourself that even if such a measure exists, it will typically not be unique. The set of measures that represent a given qualitative probability is defined by finitely many inequalities. Generically, one can expect that the set will not be a singleton.

However, Savage found that for $|X| = 2$ his relation was a qualitative probability defined on an infinite space, which also satisfied P6. This turned out to be a powerful tool. With P6 one can show that every event A can be split into two, $B \subset A$ and $A \backslash B$, such that $B \sim A \backslash B$.[7] Equipped with such a lemma, one can go on to find, for every $n \geq 1$, a partition S into 2^n equivalent events, $\Pi_n = \{A_1^n, ..., A_{2^n}^n\}$. Moreover, using P2 we can show that the union of every k events from Π_n is equivalent to the union of any other k events from the same partition. Should there be a probability measure μ that represents $\succsim$, it has to

[6] We will discuss such cognitive data in Part IV.

[7] Kopylov (2007) provides a different proof, which also generalizes Savage's theorem.

satisfy $\mu(A_i^n) = \frac{1}{2^n}$ and $\mu(\cup_{i=1}^k A_i^n) = \frac{k}{2^n}$.

Given an event B such that $S \succ B$, one may ask, for every n, what is the number k such that

$$\cup_{i=1}^{k+1} A_i^n \succ B \succsim \cup_{i=1}^k A_i^n.$$

Any candidate for a probability μ will have to satisfy

$$\frac{k+1}{2^n} > \mu(B) \geq \frac{k}{2^n}.$$

With a little bit of work one can convince oneself that there is a unique $\mu(B)$ that satisfies the above for all n. Moreover, it is easy to see that

$$B \succsim C \quad \text{implies} \quad \mu(B) \geq \mu(C). \tag{15}$$

The problem then is that the converse is not trivial. In fact, Savage provides beautiful examples of qualitative probability relations, for which there exists a unique μ satisfying (15) but not the converse direction.

Here P6 is used again. Savage shows that P6 implies that $\succsim$ (applied to events) satisfies two additional conditions, which he calls fineness and tightness. (Fineness has an Archimedean flavor, while tightness can be viewed as a continuity of sorts.) With these conditions, it can be shown that the only μ satisfying (15) satisfies also

$$B \succ C \quad \text{implies} \quad \mu(B) > \mu(B).$$

and thus represents $\succsim$ as in (14).[8]

Having established a representation of $\succsim$ by a measure, Savage's proof loses some of its dramatic effect. First, we, the audience, know that he's going to make it. In fact, he already has: restricting attention to two outcomes, and yet defining a subjective probability measure in a unique and observable way is quite a feat. Second, the rest of the proof is less exciting, though by no means trivial. Savage chose to use vNM's theorem, though this is not the only way to

[8] The examples provided by Savage also show that fineness and tightness are independent conditions. He shows a qualitative probability relation that has a unique μ satisfying (15), which is fine but not tight, and one which is tight but not fine, and neither of these has a probability that represents it as in (14).

proceed. He first shows that if two acts have the same distribution (with finite support), according to μ, they are equivalent. This means that, for a finite X, one can deal with equivalence classes defined by distributions over outcomes. Then Savage proves that the preference relation over these classes satisfies the vNM axioms, and finally he extends the representation to an infinite X.

10 Choquet Expected Utility

10.1 Capacities and Choquet Integration

Schmeidler (1989) introduced the first general-purpose, axiomatically-based model that generalized the classical ones from subjective expected utility to non-Bayesian beliefs. His model used the notion of a non-additive probability, or a capacity, which is a set function that satisfies all the conditions that probabilities satisfy, with the possible exception of additivity (but retaining monotonicity: if $A \subset B$ the capacity value of A cannot exceed that of B). Capacities were introduced by Choquet (1953-4) in a different context. He also defined a notion of integration with respect to these set functions.

Schmeidler used Anscombe-Aumann's model, and weakened the Independence axiom so that it holds only when the acts involved are "comonotonic", that is, that they move up and down – in terms of preferences – together across the states of the world. He showed that, with this restriction on the acts, the weaker axiom that results is equivalent (together with the other axioms) to an expected-utility representation, with the probability replaced by a capacity, and the standard integral – by a Choquet integral.

Formally, a *non-additive probability*, or a *capacity* on a state space S – here assumed finite – is a set function $v : 2^S \to [0,1]$ such that:

(i) $v(\varnothing) = 0$;

(ii) $A \subset B$ implies $v(A) \leq v(B)$;

(iii) $v(S) = 1$.

To define the Choquet integral of a function f with respect to a capacity v, let us start with a non-negative f. Since S is finite, we know that f takes finitely many values. Let us order them from the largest to the smallest: that is, $f = (x_1, E_1; ...; x_m, E_m)$ with $x_1 \geq x_2 \geq ... \geq x_m \geq 0$. The *Choquet integral* of f according to v is

$$\int_S f dv = \sum_{j=1}^{m} (x_j - x_{j+1}) v(\cup_{i=1}^{j} E_i) \qquad (16)$$

with the convention $x_{m+1} = 0$. If v is additive, this integral is equivalent to the Riemann integral (and to $\sum_{j=1}^{m} x_j v(E_j)$). You can also verify that (16) is equivalent to the following definition, which applies to any bounded non-negative f (even if S were infinite, as long as f were measurable with respect to the algebra on which v is defined):

$$\int_S f dv = \int_0^\infty v(f \geq t) dt$$

where the integral on the right is a standard Riemann integral. (Observe that it is well defined, because $v(f \geq t)$ is a non-increasing function of t.)

For functions that may be negative, the integral is defined so that, for every function f and constant c,

$$\int_S (f + c) dv = \int_S f dv + c$$

– a property that holds for non-negative f and c. So we make sure the property holds: given a bounded f, take a $c > 0$ such that $g = f + c \geq 0$, and define $\int_S f dv = \int_S g dv - c$.

10.2 Comonotonicity

The Choquet integral has many nice properties – it respects "shifts", namely, the addition of a constant, as well as multiplication by a positive constant. It is also continuous and monotone in the integrand. But it is not additive in general. Indeed, if we had

$$\int_S (f + g) dv = \int_S f dv + \int_S g dv$$

for every f and g, we could take $f = 1_A$ and $g = 1_B$ for disjoint A and B, and show that $v(A \cup B) = v(A) + v(B)$.

However, there are going to be pairs of functions f, g for which the Choquet integral is additive. To see this, observe that (16) can be re-written also as

$$\int_S f dv = \sum_{j=1}^{m} x_j \left[v(\cup_{i=1}^{j} E_i) - v(\cup_{i=1}^{j-1} E_i) \right].$$

Assume, without loss of generality, that E_i is a singleton. (This is possible because we only required a weak inequality $x_j \geq x_{j+1}$.) That is, there is some permutation of the states, $\pi : S \to S$, defined by the order of the x_i's, such that $\cup_{i=1}^{j} E_i$ consists of the first j states in this permutation. Given this π, define a probability vector p_π on S by $p_\pi(\cup_{i=1}^{j} E_i) = v(\cup_{i=1}^{j} E_i)$. It is therefore true that

$$\int_S f dv = \int_S f dp_\pi$$

that is, the Choquet integral of f equals the integral of f relative to some additive probability p_π. Note, however, that p_π depends on f. Since different f's have, in general, different permutations π that rank the states from high f values to low f values, the Choquet integral is not additive in general.

Assume now that two functions, f and g, happen to have the same permutation π. They will have the same p_π and then

$$\int_S f dv = \int_S f dp_\pi \quad \text{and} \quad \int_S g dv = \int_S g dp_\pi.$$

Moreover, in this case $f + g$ will also be decreasing relative to the permutation π, and

$$\int_S (f+g) dv = \int_S (f+g) dp_\pi$$

and it follows that $\int_S (f+g) dv = \int_S f dv + \int_S g dv$.

In other words, if f and g are two functions such that there exists a permutation of the states π, according to which both f and g are non-increasing, we will have additivity of the integral for f and g. When will f and g have such a permutation? It is not hard to see that a necessary and sufficient condition is the following:

f and g are *comonotonic* if there are no $s, t \in S$ such that $f(s) > f(t)$ and $g(s) < g(t)$.

10.3 Schmeidler's Axioms and Result

Schmeidler uses the Anscombe-Aumann set-up. This requires a new definition of comonotonicity, because now the acts assume values in L, rather than in $\mathbb{R}$.

For two acts $f, g \in F$, we say that f and g are *comonotonic* if there are no $s, t \in S$ such that $f(s) \succ f(t)$ and $g(s) \prec g(t)$.

Schmeidler suggested the following weakening of the Independence axiom:

Comonotonic Independence: For every pairwise comonotonic $f, g, h \in F$, and every $\alpha \in (0, 1)$,

$$f \succ g \qquad \textit{implies} \qquad \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h.$$

That is, the sole modification needed in the Anscombe-Aumann model is that now Independence is required to hold only when the acts involved are comonotonic. And the only modification in the theorem is that the probability may be non-additive:

Theorem 29 *(Schmeidler)* $\succsim$ *satisfies AA1, AA2, Comonotonic Independence, AA4, and AA5 if and only if there exist a non-additive probability measure* v *on* S *and a non-constant function* $u : X \to \mathbb{R}$ *such that, for every* $f, g \in F$

$$f \succsim g \quad \textit{iff} \quad \int_S (E_{f(s)}u)dv \geq \int_S (E_{g(s)}u)dv$$

(where the integrals are in the sense of Choquet). Furthermore, in this case v *is unique, and* u *is unique up to positive linear transformations.*

The proof is given in Schmeidler (1989, pp. 579-581, relying on Schmeidler, 1986). To understand how it works, observe that, restricting attention to acts that are constant over S, we basically have vNM lotteries, and they satisfy the vNM axioms. (Importantly, a constant act is comonotonic with any other act, and, in particular, all constant acts are pairwise comonotonic.) Thus we can find a utility function that represents preferences over lotteries, and we can plug it in. This simplifies the problem to one dealing with real-valued functions. Furthermore, if S is indeed finite we have real-valued vectors.

Now consider all vectors that are non-decreasing relative to a given permutation of the states, π. They generate a convex cone, and they are all pairwise comonotonic, so that the independence axiom holds for all three of them. Moreover, these vectors generate a mixture space – when we mix two of them, we are still inside the set. Applying Theorem 16, one gets an equivalent of Anscombe-Aumann representation, restricted to the cone of π-non-decreasing vectors. For this cone, we therefore obtain a representation by a probability vector p_π. One then proceeds to show that all these probability vectors can be described by a single non-additive measure v.

11 Maxmin Expected Utility

11.1 Model and Theorem

This model builds on Anscombe-Aumann. It weakens the Independence axiom and introduces two axioms, each of which is strictly weaker than the Independence axiom. They are:

C-Independence: For every $f, g \in F$, every constant $h \in F$ and every $\alpha \in (0,1)$,

$$f \succ g \quad \textit{implies} \quad \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h.$$

Uncertainty Aversion: For every $f, g \in F$, if $f \sim g$, then, for every $\alpha \in (0,1)$,[9]

$$\alpha f + (1-\alpha)g \succsim f.$$

Thus, uncertainty aversion requires that the decision maker have a preference for mixing. Two equivalent acts can only improve by mixing, or "hedging" between them. Observe that uncertainty aversion is also a weakened version of Anscombe-Aumann's independence axiom (which would have required $\alpha f + (1-\alpha)g \sim f$ whenever $f \sim g$).

Theorem 30 *(Gilboa and Schmeidler, 1989) $\succsim$ satisfies AA1, AA2, C-Independence, AA4, AA5, and Uncertainty Aversion if and only if there exist a closed and convex set of probabilities on S, $C \subset \Delta(S)$, and a non-constant function $u : X \to \mathbb{R}$ such that, for every $f, g \in F$*

$$f \succsim g \quad \textit{iff} \quad \min_{p \in C} \int_S (E_{f(s)}u)dp \geq \min_{p \in C} \int_S (E_{g(s)}u)dp$$

Furthermore, in this case C is unique, and u is unique up to positive linear transformations.

The uniqueness of C is relative to the conditions stated. Explicitly, if there is another set C' and a utility function u' that satisfy the above representation, $C' = C$ and u' is a pat of u.

[9] It suffices to require this condition for $\alpha = 1/2$.

11.2 Idea of Proof

One would first define the function $J : \mathbb{R}^n \to \mathbb{R}$ so that

$$f \succsim g \quad \Leftrightarrow \quad J(f) \geq J(g)$$

by letting

$$J((c, c, ..., c)) = c$$

for each $c \in \mathbb{R}$, and setting, for each f,

$$J(f) = c$$

for c such that $f \sim (c, c, ..., c)$.

The Uncertainty Aversion axiom guarantees that J is concave.

Next, let a_f be the coefficients of the supporting hyperplane at f, so that

$$\begin{aligned} a_f f &= J(f) \\ a_f g &\geq J(g) \quad \forall g \\ &\text{and} \\ J(f) &= \min_g a_g f \end{aligned}$$

Consider a specific f and its supporting hyperplane a_f. Assume that $J(f) = c$ so that $f \sim (c, c, ..., c)$ and $a_f f = c$. Axiom C-independence implies that

$$f \sim \alpha f + (1 - \alpha)(c, c, ..., c) \sim (c, c, ..., c)$$

for all $\alpha \in [0, 1]$. This implies that the supporting hyperplane defined by a_f has to coincide with J on the segment connecting f and $(c, c, ..., c)$. Hence

$$a_f (c, c, ..., c) = c$$

and the latter means that $a_f 1 = 1$. Coupled with the non-negativity of a_f, we conclude that it is a probability vector.

12 Arrow's Impossibility Theorem

Assume that there is a set of alternatives $A = \{1, 2, ..., m\}$ with $m \geq 3$ and a set of individuals $N = \{1, 2, ..., n\}$ with $n \geq 2$.

Let the set of linear orderings be $R = \{\succ \subset A \times A | \succ$ complete, transitive, a-symmetric$\}$

A preference aggregation function maps profiles of preferences to a preference that is attributed to society. That is, a *preference aggregation function* is $f : R^n \to R$.

Given such a function, define:

1. *Unanimity*: For all $a, b \in A$, if $a \succ_i b \; \forall i \in N$, then $af((\succ_i)_i)b$

2. *Independence of Irrelevant Alternatives (IIA)*: For all $a, b \in A$, $(\succ_i)_i, (\succ'_i)_i$ if

$$a \succ_i b \Leftrightarrow a \succ'_i b \qquad \forall i \in N$$

then

$$af((\succ_i)_i)b \Leftrightarrow af((\succ'_i)_i)b.$$

An aggregation function is *dictatorial* if there exists $i \in N$ such that, for all $(\succ_j)_j$.

$$f((\succ_j)_j) = \succ_i$$

Theorem 31 *(Arrow, 1951) f satisfies Unanimity and IIA iff it is dictatorial.*

Proof: Clearly, all (the n different) dictatorial functions satisfy the two conditions. The interesting (in fact, amazing) fact is that the opposite it true as well. We turn to prove it now (based on one of the short proofs provided by Geanakoplos, 2005).

Step 1: For every $a \in A$ and $(\succ_i)_i$, if a is extreme (top or bottom) in each of $\succ_i$, then it is extreme in $\succ = f((\succ_i)_i)$.

Proof:

Assume not. Then, there exists a profile $(\succ_i)_i$ and an alternative a such that a is extreme in each of $\succ_i$ but not in $\succ$. Thus, there are $b, c \in A$ such that $b \succ a \succ c$. We can modify the profile $(\succ_i)_i$ to get another profile $(\succ_i')_i$ such that

(i) a is top (bottom) at $\succ_i'$ whenever it is top (bottom) at $\succ_i$;

(ii) $c \succ_i b$ for all i

– simply by switching between b and c, if needed, in the profile $\succ_i$.

The ranking between a and any other alternative has not changed (it is the same in $\succ_i'$ as in $\succ_i$ for each i), and, by IIA, a is ranked, relative to b and c, in $\succ' = f((\succ_i')_i$ as it was in $\succ = f((\succ_i)_i)$. Thus, $b \succ' a \succ' c$ while unanimity implies $c \succ b$. $\square$

Step 2: There exists $i = i(a) \in N$ such that

$$b \succ_i c \Rightarrow b \succ c \qquad \forall b, c \in A \backslash \{a\}$$

Proof: To find i, consider profiles such that a is extreme according to each individual. We can start with a being at the bottom of the ranking $\succ_i$ for each i, and switch a to the top, one individual at a time. In all of these, a should be extreme in $\succ = f((\succ_i)_i)$, where, by unanimity, it has to be at the bottom at the beginning, and at the top at the end.

Let i be the individual for which the first jump occurs from bottom to top of $\succ$. It follows (using IIA again), that for all $d \neq a$,

$$\text{(I)} \quad \begin{array}{ccc} d & \succ_j a & \forall j \leq i \\ a & \succ_j d & \forall j > i \end{array}$$

$$\Rightarrow d \succ a$$

and

$$\text{(II)} \quad \begin{array}{ccc} d & \succ_j a & \forall j < i \\ a & \succ_j d & \forall j \geq i \end{array}$$

$$\Rightarrow a \succ d.$$

Given distinct $b, c \in A \backslash \{a\}$, consider

$$b \succ_i a \succ_i c$$

$$\begin{aligned} d &\succ {}_j a \qquad \forall d \neq a, j < i \\ a &\succ {}_j d \qquad \forall d \neq a, j > i. \end{aligned}$$

Then on $\{b, a\}$ preferences look like pattern (I) and $b \succ a$ follows. On $\{c, a\}$ preferences look like pattern (II) and $a \succ c$ follows. Hence $b \succ c$. Finally, due to the IIA, this has to be the case whenever the individuals have the same rankings between b and c as in such profiles. However, the b/c rankings of the other individuals were not constrained above, which means that individual i's ranking between b and c determine that of society's. □

Step 3: $i(a)$ is independent of a.

Proof: If not, there are a, b with $i(a) \neq i(b)$. Consider $c \neq a, b$. Let

$$\begin{aligned} a &\succ {}_{i(c)} b \\ b &\succ {}_{i(a)} c \\ c &\succ {}_{i(b)} a \end{aligned} \tag{17}$$

which is possible unless $i(a) = i(c) = i(b)$. However, in this case society's preferences would be cyclical. Hence, it has to be the case that there is no profile for which (17) happens. That is, it has to be the case that $i(a) = i(c) = i(b)$ and the conclusion follows. □

Step 4: i is a dictator. Given $a, b \in A$, choose $c \neq a, b$ and use $i = i\,(c)$. □

A similar result, with a similar proof, applies to weak orders if unanimity is strengthened to weak and strong unanimity.

13 References

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