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INFORMATION DEPENDENT GAMES
Can Common Sense be Common Knowledge?

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This paper attempts to study the consistency of several basic game-theoretic axioms. Two by-products are the introduction of information-dependent games, and a formal treatment of the framework of game theoretic axioms. In this setup a version of the Surprise Test Paradox is used to prove that common sense cannot be common knowledge.

1. Introduction

Axiomatic derivations of solutions to non-cooperative games have recently become a central topic of research in game theory. The main solution concepts introduced in the previous decade were axiomatized in the last few years: correlated equilibria [Aumann (1974)] have been axiomatized in Aumann (1987). Brandenburger and Dekel (1986, 1987) modified Aumann's framework [more specifically, they diverted from von Neumann–Morgenstern (1944) expected utility theory) to characterize Selten's (1975) perfect equilibrium and its variant proper equilibrium [Myerson (1978)].

Another modification of Aumann's axioms (the rejection of the common prior assumption), carried out by Tan and Werlang (1985), resulted in an axiomatization of rationalizable equilibria [introduced by Bernheim (1984) and Pearce (1984)]. [1]

The two main axioms introduced by Aumann are Bayesian rationality and common knowledge. The first one is a behavioristic axiom, stating that each player will maximize her expected utility given her Bayesian posterior. This is, in fact, the 'best response' notion inherent in the concept and the definition of Nash equilibrium.

Common knowledge is the more revolutionary axiom, which is essentially a meta-axiom. Originally introduced into game theory in Aumann (1976), it may be phrased as: The axioms of logic, the axioms of game theory, the behavioristic axioms and the game itself – are all common knowledge. This statement of the axiom is implicit in most models of non-cooperative games. In Gilboa (1986) it is shown how it can be made explicit, and that in this case it implies that this axiom itself – and thus the whole model – is common knowledge. [For alternative formalization see Kaneko (1987)].

Our axiom of 'common sense' appearing in the title is a weakening of Aumann's Bayesian rationality, and, as a matter of fact, seems to be one of the weakest concepts of rationality one can think of: it says that if a player in a game has a unique dominant strategy, he will choose it.

In this note we examine the behavioristic axiom of common sense and the meta-axiom of common knowledge, when applied to the wider framework of information-dependent games which will be

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[1] Kohlberg and Mertens' (1987) treatment of refinement of Nash equilibria, using the behavioristic assumptions of backward and forward induction, may also be considered axiomatic.

introduced by examples in section 2. The formal treatment can be found in section 3 and concluding remarks in section 4. Our main conclusion is that, in this framework, common sense cannot be in general common knowledge.

2. Information-dependent games

Information-dependent games are non-cooperative games in which the player's payoff varies with his or other players' information about the play of the game. Let us first consider several examples:

2.1. Example: Catch 22

In Joseph Heller's book (1955), Yossarian could be released from combat duty only if he applied for release on grounds of insanity and the army medical authorities agreed with him, whereas it was obvious that carrying on with the combat duties was indeed insane, the catch was that applying for release proved sanity.

One way to model this situation is as a two-person game between Yossarian and the army. Yossarian has two moves (strategies), namely to apply or not to apply for release. The army is a dummy player (in the sense of having a single strategy). If the army doesn't know for sure that Yossarian is *not* going to apply, then insanity cannot be proved and Yossarian's payoff will be zero. However, if the army does know that he is not going to apply – hence insanity is proved – then his payoff will be 1 if he applies and zero otherwise.

2.2. Example: The sweet revenge

Mr. A hates Mr. B for 'wrongs' B did to him. Assuming Mr. A can follow a strategy which will hurt B, A will choose such a strategy for the sake of revenge. However, the revenge will be sweeter if A knows that B anticipates it. That is to say, the same play of the game may result in a higher payoff for A if B *knows* that this indeed is going to be the play of the game.

2.3. Example: Gossip

Gossip is a phenomenon where the utility depends not only upon the information learnt or told, but also on who else knew it first. Here again, the very same actions may induce different payoffs – according to the information the players have.

2.4. Example: Fashion

Fashion is considered to be an example of changes tastes. However, most of the consumers do not really change their inherent tastes but rather try to conform to the views of others. [See Karni and Schmeidler (1987)]. Once more, utility depends on information: the utility of wearing a certain tie depends on whether or not others know that Prince Charles wears the same (a similar?) tie.

The next example is essential to our analysis.

2.5. Example: The surprise test

Player 1, the teacher, has to give a 'surprise' test to his class, player 2, on one of two days, say Thursday or Friday. His payoff is 1 if he succeeds in surprising his class, i.e. (by definition), if the evening before the tests the class could not have known for sure that indeed it was going to take place the next day. His payoff is 0 otherwise.

A smart student argues as follows: if the test will be on Friday, the class will know it on Thursday evening because there was no test on Thursday. Assuming that the teacher knows this, he won't give the test on Friday, but rather will try to surprise the class on Thursday. However, if the class knows that the teacher has common sense, it will not be surprised on Thursday, either. Hence there is no way to give the surprise test to the class.

The story goes on that the class was completely surprised to be surprised, i.e., to be given the surprise test on Thursday.

This example proves our point. [2,3]

3. Model and results

An information-dependent game is a quadruple $G=(N,(S^i)_{i\in N}, K, (h^i)_{i\in N})$ where $N=\{1,\dots,n\}$ $(n\geq 1)$ is the test of *players*; S^i is a non-empty set of *strategies* of player i; [4] $\phi \neq K \subset (2^S\setminus\{\phi\})^n$ where $S=\prod_{i=1}^n S^i$ is the set of possibe *prediction profiles*; and h^i: $S\times K \to \mathscr{R}$ is i's *payoff function*.

An element of K, say $(V_1,\dots,V_n)$, should be interpreted as follows: for each i, player i predicts [believes, expects, anticipates, knows(?)] that the combination of strategies $s\in S$ which will be eventually played (the play of the game) belong to $V_i \subset S$.

Let us consider an example:

$$N=\{1,2\},\qquad S^1=\{0,\ L\},\quad S^2=\{0,\ L\},$$

$$K=\{(S,\ S),((0,0),(0,0))\},$$

and h^i are given by the following two matrices, one for each possible profile of prediction (the bold line encloses the prediction sets V_1 and V_2, which are identical in this case).

	O	L
O	2, 2	0, 1
L	1, 0	5, 5

	O	L
O	2, 2	0, −10
L	−10, 0	−10, −10

The story behind this game is the following: Player 1 and player 2 have an appointment. Each one of them may be either On time, O, or Late, L.

2 Economic variants of the 'surprise test' example are those of unanticipated inflation and unanticipated devaluation. In both these cases, government's policy is successful only if it is not predicted by the public. However, as opposed to the surprise test, these situations are not paradoxical, since inflation (devaluation), when anticipated, becomes a dominant strategy.

3 A slightly different example, in which a player's payoff depends on his action as well as on the belief of other players, is suggested by the nature of 'supreme being' as portrayed by Brams (1983): The supreme being derives utility from being believed in only if it does not reveal itself, although revelation induces belief. (In Brams' representation, belief is a matter of strategic choice.)

4 Note that $\{S^i\}_i$ may be infinite; in particular, they may be the sets of mixed strategies in a game with finitely-many pure strategies.

The case $V=(S, S) \in K$ describes a situation in which neither of the players knows whether or not the other one is going to be on time. The payoffs are self-explanatory. On the other hand, the profile $V'=(\{(0, 0)\}, \{(0, 0)\}) \in K$ describes the situation in which both players predict that they are going to be on time. It is typical of self-enforcing social norms (punctuality in this case): the existence of the norm causes disutility to its violators (hence the payoff of -10 to a latecomer). This payoff, in turn, reinforces the norm. Abiding by it is a dominant strategy.

In order to define formally the concept of *informationally consistent play*, we explicitly refer to game theoretic axioms as mathematical objects. Let $\mathscr{A}$ be a set of axioms, including the axioms of logic and set theory necessary for standard mathematics, as well as a set of axioms describing a model of a game with information and meta-information. For the latter, one may use a model of information as in Gilboa (1986) or Kaneko (1987) – or other models of modal logic and knowledge as Kripke (1963) or Hintikka (1962) [see Halpern (1986) for further references] – and add to it an axiom defining the game (as a problem) and its play (as a solution). For instance, one may write down an axiom stating that there exists a function from the set of states of the world onto the set of all pairs (G, s) where G is a (standard) game (in normal form) and s is a possible play of G. All axioms mentioned so far will be called 'the model axioms'. $\mathscr{A}$ may also contain (and usually does) some game theoretic solution axioms. Examples of such axioms are:

CK (common knowledge): The game and all the other axioms in $\mathscr{A}$ *are common knowledge.*

CS (common sense): If a player has a dominant strategy, she will play it.

A more formal way to describe this axiom, as well as other behavioral axioms, will be of the type: If G belongs to a certain class of games, then its play belongs to a certain class of possible plays.

Note that this is a very weak axiom of 'rational' behavior. For example, the axiom stating that a player will not play a strictly dominated strategy implies CS but the converse is false.

A stronger but prevalent axiom is that of best response, inherent in the concept of Nash equilibrium and its various variants. For example, a version of this axiom has been stated in Aumann (1987) under the name 'Bayesian rationality'. Other variants may be proposed for other equilibrium concepts such as Nash and perfect equilibria as well as sequential equilibrium [see Kreps and Wilson (1982)].

Different kinds of axioms are those dealing with invariance properties of the solutions (more precisely, covariance), e.g., under permutations of names of the players and/or the strategies and of affine transformations of utilities. Axioms of invariance of the solution with respect to the representation of the game may be found in Kohlberg and Mertens (1986). (They introduce other axioms as well.) See also Mertens (1987) for another invariance axiom.

We are now in a position to define informationally consistent plays. Given an information-dependent game $G=(N, (S^i)_{i\in N}, K, (h^i)_{i\in N})$ and a set of axioms $\mathscr{A}$, a pair $(s, (V_1,\dots,V_n)) \in S \times K$ is an informationally consistent play if

(i) $s \in \bigcap_{i=1}^{n} V_i$;
(ii) For each $i \in N$, the axioms $\mathscr{A}$ and the assumption that the game is $G'=(N, (S^i)_{i\in N}, (h^i(\cdot, V_1, V_2,\dots,V_n))_{i\in N})$ imply that i knows that the play of the game will be in V_i, and V_i is the minimal subset of S satisfying this requirement.

In the punctuality game introduced earlier, if the only game-theoretic assumptions in $\mathscr{A}$ are CS and CK, then there are five informationally-consistent plays. [Since there are two profiles $V=(V_1, V_2) \in K$, and for each of them V_i is provable given A and $h(\cdot, V)$.] However, if we add the axiom of best response in the sense of Nash (for pure strategies) to $\mathscr{A}$, only one informationally-consistent play is left. [$s = (0, 0)$ and $V_1 = V_2 = \{(0, 0)\}$.] The information profile $V_1 = V_2 = S$ is ruled out by this axiom since not all the plays in S are pure-strategy Nash equilibria.

We shall now phrase two trivial results.

Proposition 1. *Let $(s, (V_1, \ldots, V_n))$ be an informationally-consistent play of the information-dependent game G with respect to $\mathscr{A}$. If $CK \in \mathscr{A}$, then $V_1 = V_2 = \ldots = V_n$.*

Proof. Trivial since the axiom CK ensures that all players will be able to prove the very same propositions. □

An information dependent game is full if $\{(V_1, \ldots, V_n) \mid V_1 = V_2 = \ldots = V_n \neq \emptyset,\ V_1 \in 2^S\} \subset K$.

Proposition 2. *Assume CK, CS are the only game theoretic solution axioms in $\mathscr{A}$. Then there exists a full information-dependent game which has no informationally-consistent play.*

Proof. Consider the surprise test example (2.5 above). It can formally represented by the following information-dependent game $G: N = \{1, 2\}$, $S^1 = \{T, F\}$, $S^2 = \{s\}$, $K = \{(S, S), (\{(T, s)\}, \{(T, s)\}), (\{(F, s)\}, \{(F, s)\})\}$ and h^i are given by the following matrices:

	s
T	1, 0
F	0, 0

$V_1 = V_2 = S$

	s
T	0, 0
F	0, 0

$V_1 = V_1 = \{(T, s)\}$

	s
T	1, 0
F	0, 0

$V_1 = V_2 = \{(F, s)\}$

(For simplicity we assume the class' payoff to be identically zero; in view of Proposition 1 we assume K includes only elements $V = (V_1, V_2)$ such that $V_1 = V_2$.)

It is easily seen that there is no informationally-consistent play for G. □

4. Discussion

(a) It should be pointed out that if the game does not depend upon the information, then the axioms of common sense and common knowledge are perfectly consistent.

One can state (and prove) a result similar to Proposition 2, which says that the axioms of best response and common knowledge are inconsistent. Such a result – in the framework of pure strategies as in the previous section – would hardly be surprising. Indeed, for any information-independent game without pure Nash equilibria, there would be no informationally-consistent plays. However, the results would still hold in the obvious extension to mixed strategies (where S^i consists of the *mixed* strategies of player i).

(b) Intuitively one would expect that the addition of other game-theoretic axioms to $\mathscr{A}$ would not change the conclusion of Proposition 2. However, it turns out to be a fallacy: Firstly, note that our proof fails. Consider the following axioms: 'If a player has a strategy denoted by the symbol 'T' – he would play it'. If this axiom is added to $\mathscr{A}$, then there exists an informationally-consistent play to the game: The teacher gives the test on Thursday, and everybody can prove it using this new axiom. Secondly, consider the axiom: 'In every game, every player's strategies are enumerated. Every player will play his first strategy unless he has a unique dominant strategy'.

On the other hand, if game-theoretic axioms in $\mathscr{A}$ were required to satisfy certain conditions (meta-axioms), such as symmetry of strategies with respect to payoff, Proposition 2 could indeed be extended.

(c) One can think of an obvious extension of the concept of information-dependent game as defined above: The payoffs may depend not only upon what the players know, but also on higher orders of knowledge, i.e., what the players know about what other players know about what will be the play of the game, etc.

(d) Game theoretic axioms are usually used to derive information (about the play of the game) from the payoffs. The concept of information-dependent games allows us deductions in the opposite direction, from information to payoffs, as well. This circularity may result in inconsistencies, as exemplified in Proposition 2. Our point is that the information-to-payoff direction appears naturally in modeling the situation, hence a satisfactory resolution of the inconsistency cannot be a dismissal of the new concept of information-dependent games. One is therefore led to a more critical appraisal of the game theoretic axioms. Our instincts point out to the CK axiom as the villain.

(e) The axiom of CK in its full strength is not needed for Proposition 2. Indeed, only second-order knowledge is used in the proof. Hence our impossibility result cannot be dismissed on the grounds of bounded rationality of the players. [Bounded rationality sometimes excludes the applicability of the CK axiom. See Halpern (1986) and the references there.]

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