# A Model of Modeling

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## Outline

- Four distinctions:
  - Theories and paradigms
  - Rationality
  - Models in decision theory
  - Models in economics
- The common structure

## Theories and Paradigms

- Theories – concrete concepts:
  - growth, inequality, price...
- Paradigms – more flexible concepts:
  - Decision under certainty: "alternative"
  - Under uncertainty: "state", "outcome"
  - Game theory: "player", "strategy"...

## Are Decision/Game Theory Refutable?

- Examples
  - The Ultimatum Game
  - Are there framing effects?
  - Can consequentialism be violated?
- "Conceptual Frameworks" (Gilboa and Schmeidler, 2001)
- "Theories are not refuted, they are embarrassed" (Amos Tversky)

Two Notions of Rationality

- Objective: one can convince “any reasonable person” that one is right
- Subjective: it is not the case that “any reasonable person” would be convinced that one is wrong

## Two Ways of Using Decision Models

- Classical OR: one models the problem (variables, objective, constraints) and gets the optimal solution from the software.
  - Example: Google maps
- Mere consistency check: one tests whether one's intuition makes sense; or what does it take to justify it
  - Example: Investment; Emigration; Job
- And a whole gamut (of a dialog) in between

## Two Ways of Thinking about Economics

- Classical: a science that should make predictions
  - Whether in a Popperian, rule-based way
  - or in a case-based way (refutable with specified similarity)
- Alternative: criticism of reasoning; testing whether arguments makes sense
  - logically (mathematics)
  - economically (equilibrium analysis...)
  - empirically (empirical and experimental work...)

Our goal

- To offer a formal model of the act of modeling
- In which we can capture one aspect of each of the four distinctions
- See the analogies between them
- Prove some results

Example 1: Decision under Certainty

- $Act\,(a)$ means that $a \in A$
- $\succsim (a, b)$ means that $(a, b) \in \succsim$
- The theory takes a set of statements and augments it
- Say, to $\{\succsim (a, b), \succsim (b, c)\}$ add $\{\succsim (a, c)\}$

Example 2: Game Theory

- Battle of the Sexes:
- $Player\,(P1)\,,\,Player\,(P2)$

$Act\,(P1, A1)\,,\,Act\,(P1, B1)$
$Act\,(P2, A2)\,,\,Act\,(P2, B2)$
$Outcome\,(o_1)\,,\,Outcome\,(o_2), Outcome\,(o_3)\,,\,Outcome\,(o_4)$
$Result\,(A1, B1, o_1)\,,\,Result\,(A1, B2, o_2)$
$Result\,(A2, B1, o_3)\,,\,Result\,(A2, B2, o_4)$
$\succsim (P1, o_1, o_4)\,,\succsim (P1, o_4, o_2), \succsim (P1, o_2, o_3)\,,\succsim (P1, o_3, o_2)\,, \ldots$
$\succsim (P2, o_4, o_1)\,,\succsim (P2, o_1, o_2), \succsim (P2, o_2, o_3)\,,\succsim (P2, o_3, o_2)\,, \ldots$

- Add

$May\,(o_1)\,,\,May\,(o_4)$

Descriptions

- Entities $E \subset \hat{E}$ ($\hat{E}$ infinite)
- Predicates:
  - a $k$-place predicate $f : E^k \to \{0, 1\}$
  - More convenient to define
  - $\circ$ – irrelevant; $*$ – unknown

$$X = X_0 \cup \{\circ, *\}$$

$$\mathcal{E} = \cup_{k \geq 1} E^k,$$

  a *description*

$$d : F \times \mathcal{E} \to X$$

  - the *degree* of $f$ according to $d$: $m$ such that $\forall\ k \neq m$, $\forall e \in E^k$, $d\,(f, e) = \circ$.

$D = D(F, E)$ be the set of all descriptions for the set of entities $E$ and predicates $F$.

Example: The Dictator Game

- $E = \{P1, P2, 0, ...,100, (100; 0), ..., (0, 100)\}$
- $F = \{Player, Act, Outcome, Result, \succsim, May\}$
- Values of predicates given by the description:
    - $d(Player, (P1)) = 1, d(Act, (P1)) = 0,$
    - $d(Player, (0)) = 0, d(Act, (0)) = 1, ...,$
    - $d(Player, (P1, 0)) = \circ, ...$
    - $d(\succsim, (P1, (100; 0), (99; 1))) = 1,$
    - $d(\succsim, (P1, (99; 1), (100; 0))) = 0, ...,$
    - $d(\succsim, (P1, (99; 1))) = \circ, ...$

## Compatibility of Descriptions

- Two descriptions $d, d' \in D$ are *compatible* if:
  - (i) for every $f \in F$ and every $e \in \mathcal{E}$,

$$d(f, e) = \circ \qquad \Leftrightarrow \qquad d'(f, e) = \circ$$

  - (ii) for every $f \in F$, every $e \in \mathcal{E}$, and every $x, y \in X_0$, if

$$d(f, e) = x \qquad \textit{and} \qquad d'(f, e) = y$$

  then $x = y$.

## Extension of a Description

- $(d, d' \in D)$ $d'$ is an *extension* of $d$, denoted $d' \rhd d$, if:
  - (i) for every $f \in F$ and every $e \in \mathcal{E}$,

$$d(f,e) = \circ \qquad \Leftrightarrow \qquad d'(f,e) = \circ$$

  - (ii) for every $f \in F$, every $e \in \mathcal{E}$, and every $x \in X_0$,

$$d(f,e) = x \qquad \Rightarrow \qquad d'(f,e) = x.$$

- Clearly, if $d' \rhd d$ then $d$ and $d'$ are compatible.

## Reality

| | Reality | Representation |
|---|---|---|
| Language | $(F_R, E_R)$ | $(F, E)$ |
| Input, or question | $d_R$ | $d$ |
| Output, or answer | $d'_R$ | $d'$ |

.

Figure: We model reality as a set of entities $E_R$ and predicates $F_R$, with $d_R$ characterizing what is known and an extension $d'_R$ characterizing a possible output of answer.

## Models

- Formally, a *model* is a quintuple $M = (F_R, E_R, F, E, \phi = \phi_F \cup \phi_E)$ such that:
  - $F_R \subset \hat{F}_R$; $E_R \subset \hat{E}_R$; $F \subset \hat{F}$; $E \subset \hat{E}$
  - $\phi_F : F_R \to F$ is a bijection
  - $\phi_E : E_R \to E$ is a bijection.
- Acceptable models: given $d_R \in D(F_R, E_R)$, and a set of bijections

$$\Phi_F \subset \left\{ \phi_F \,\middle|\, \begin{array}{l} \phi_F : F_R \to F, \\ \phi_F \text{ is a bijection} \end{array} \right\}.$$

The set of acceptable models $\Phi(F_R, E_R)$ consists of the bijections obtained by the union of a $\phi_F : F_R \to F$ in $\Phi_F$ and any bijection $\phi_E : E_R \to E$.

Theories

- Given a set of descriptions $D = D(F, E)$, a *theory* is a function

$$T : D \to D$$

such that, for all $d \in D$,

$$T\left(d\right) \triangleright d$$

Using Theories to Examine Reality

- Given
  - a model $M = (F_R, E_R, F, E, \phi = \phi_F \cup \phi_E)$,
  - a description of reality, $d_R \in D(F_R, E_R)$,
  - and a theory $T$,
- define the $M$-$T$-extension of $d_R$, $d'_R(d_R, M, T)$ as follows:
  - (i) define a description $d \in D(F, E)$ by $d(f, e) = d_R\left(\phi^{-1}(f), \phi^{-1}(e)\right)$ for all $(f, e) \in F \times E$; denote this description by $d_R \circ \phi^{-1}$;
  - (ii) apply $T$ to obtain an extension of $d$, $d' = T(d)$
  - (iii) define a description $d'_R = d'_R(d_R, M, T) \in D(F_R, E_R)$ by $d'_R(f, e) = d'(\phi(f), \phi(e))$ for all $(f, e) \in F_R \times E_R$; denote this description $d' \circ \phi$.

Theories and Reality

Figure 2 illustrates a trivial but important point given by the following.

| | Reality | | Representation | |
|---|---|---|---|---|
| Language | $(F_R, E_R)$ | | $(F, E)$ | |
| Input, or question | $d_R$ | $\xrightarrow{\phi}$ | $d$ | $= d_R \circ \phi^{-1}$ |
| | | | $\downarrow$ | $T$ |
| Output, or answer | $d'_R$ | $\xleftarrow{\phi^{-1}}$ | $d'$ | $= T(d) = T$ |

Figure: An illustration of how a theory is used to draw conclusions about reality. Beginning with a description of reality $d_R$, we use the model $M$ to construct the formal description $d$ satisfying $d(f, e) = d_R(\phi^{-1}(f), \phi^{-1}(e))$. The theory $T$ then gives the extension $T(d)$, at which point we again use the model to construct the description of reality given by $d'_R = d'_R(d_R, M, T) = T\left(d_R \circ \phi^{-1}\right) \circ \phi$. $d'_R$ is the $M$-$T$-extension of $d_R$.

## A Simple Observation

For every model $M = (F_R, E_R, F, E, \phi)$, description $d_R \in D(F_R, E_R)$, and theory $T$, the $M$-$T$-extension of $d_R$,

$$d'_R(d_R, M, T) = T\left(d_R \circ \phi^{-1}\right) \circ \phi \in D(E_R, E_R)$$

is an extension of $d_R$.

## Compatibility for a Given Model

- Given $d''_R$ (new data...)

| | | Reality | | Representation | |
|---|---|---|---|---|---|
| Language | | $(F_R, E_R)$ | | $(F, E)$ | |
| Input, or question | | $d_R$ | $\xrightarrow{\phi}$ | $d$ | $= d_R \circ \phi^{-1}$ |
| | $N$ | $\downarrow$ | | $\downarrow$ | $T$ |
| | | $d''_R$ | | | |
| Output, or answer | | $d'_R$ | $\xleftarrow{\phi^{-1}}$ | $d'$ | $= T(d) =$ |

Figure: Illustration of how data, normative considerations, or other considerations are used to evaluate a theory. Given a description $d_R$, the model $M$ and theory $T$ are used to construct its $M$-$T$-extension $d'_R$, as described in Figure 2. The process $N$, perhaps representing the collection of additional data, generates the extension $d''_R$. We say that the theory $T$ is weakly compatible with $d''_R$ if there is

Compatibility

- Given a description of reality $d_R \in D(F_R, E_R)$, an extension thereof $d_R''$, and a theory $T$,
  - $T$ is *strongly compatible* with $d_R''$ if it is $M$-compatible with $d_R''$ for every model $M = (F_R, E_R, F, E, \phi)$ derived from an acceptable $\phi \in \Phi$.
  - $T$ is *weakly compatible* with $d_R''$ if it is $M$-compatible for at least one such model $M$.

## Necessitation for a Given Model

- For a given $(f, e) \in F_R \times E_R$ such that $d_R(f, e) = *$, and a value $x \in X_0$, define $d''_R(f, e, x)$ by the minimal extension of $d_R$ such that $d''_R(f, e) = x$. A theory $T$ *M-necessitates* $(f, e, x)$ if $d'_R$ is an extension of $d''_R(f, e, x)$.

## Necessitation

- Given a description of reality $d_R \in D(F_R, E_R)$, a pair $(f, e) \in E_R \times F_R$ such that $d_R(f, e) = *$, a value $x \in X_0$, a conclusion $(f, e, x)$ and a theory $T$, we say that
  - $T$ *strongly necessitates* $(f, e, x)$ if, for every $\phi \in \Phi$, $T$ $M$-necessitates $(f, e, x)$ for $M = (F_R, E_R, F, E, \phi)$.
  - $T$ *weakly necessitates* $(f, e, x)$ if, there exists $\phi \in \Phi$, such that $T$ $M$-necessitates $(f, e, x)$ (for $M$ as above).

## Complexity Results

### Proposition

*Given a description of reality* $d_R \in D(F_R, E_R)$*, a pair* $(f, e) \in F_R \times E_R$ *such that* $d_R(f, e) = *$*, a value* $x \in X_0$*, a conclusion* $(f, e, x)$*, a theory* $T$*, and a set* $\Phi$*, it is NP-Hard to determine whether* $T$ *weakly necessitates* $(f, e, x)$*.*

Next we show that a similar conclusion applies to strong necessitation.

### Proposition

*Given a description of reality* $d_R \in D(F_R, E_R)$*, a pair* $(f, e) \in E_R \times F_R$ *such that* $d_R(f, e) = *$*, a value* $x \in X_0$*, a conclusion* $(f, e, x)$*, a theory* $T$*, and a set* $\Phi$*, it is NP-Hard to determine whether* $T$ *strongly necessitates* $(f, e, x)$*.*

## Conclusion

- The distinction between strong ($\forall$) and weak ($\exists$) necessitation captures some of the distinctions between:
  - Refutations of theories vs. conceptual frameworks
  - Judgments of objective vs. subjective Rationality
  - Uses of decision theory models
  - Roles of economics
- The discussion of economics, game and decision theory can be more focused if we better understand the act of modeling and the degree of freedom involved.