No-Betting Pareto Dominance

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I. Introduction

I.1 Trade

- Suppose Alice and Bob have one banana and one mango each.
- Their utility functions are linear. Alice is indifferent between 1 unit of banana and 2 units of mango, and Bob is indifferent between 2 units of banana and 1 unit of mango.
- In the competitive equilibrium of this economy, Alice consumes only bananas and Bob consumes only mangos.

I.2 Speculation

- Cathy and Doug have one dollar each.
- There are two states. Cathy thinks state 1 has probability 2/3, Doug thinks state 1 has probability 1/3. They are both risk neutral.
- In the competitive equilibrium of this economy, Cathy has no money in state 2 and Doug has no money in state 1.

I.3 Speculation Again

- Ellen has an idea for a startup, worth $\$10,000,000$ if successful, but requiring $\$100,000$ in funding from Frank.
- Ellen thinks the probability of success is $9/10$. Frank thinks the probability of success is $1/10$. They are both risk neutral.
- Despite his relative pessimism, Frank is willing to invest.

I.4 Examples 1 and 2, The Common Elements

– Consider an Arrow-Debreu economy with two goods and two agents, with utility functions

$$\begin{aligned} u_A(x_1, x_2) &= \frac{2}{3}x_1 + \frac{1}{3}x_2 \\ u_B(x_1, x_2) &= \frac{1}{3}x_1 + \frac{2}{3}x_2 \end{aligned}$$

and endowments $e_A = e_B = (1, 1)$.

– The competitive equilibrium price vector is $(1, 1)$ and allocations are $(2, 0)$ for A and $(0, 2)$ for B. The outcome is Pareto efficient.

I.5 But ...

- Trading to achieve a Pareto efficient outcome seems compelling in Example 1, where gains are based on differences in taste.

- Trading to achieve a Pareto efficient outcome seems less compelling in Example 2, where gains are based on differences in beliefs. Indeed, no common belief could generate gains from trade in this case.

- Beliefs again differ in Example 3, but here there are common beliefs (including the beliefs of each agent) that would generate gains from trade.

I.6 This Paper ...

- Presents a new, weaker (than Pareto) dominance relation, "No-Betting Pareto" dominance.
- This relation allows more allocations to be undominated. In particular, we will not accept a dominance relationship that depends too heavily on different beliefs.
- We relate our notion of "too heavily" to a willingness to bet.
- This notion pushes us in the direction of weakening the argument for free financial markets.

II. The Model

II.1 The Elements

- Agents $N = \{1, ..., n\}$
- A measurable state space (S, Σ)
- Outcomes X
- Simple acts: $F = \left\{ f : S \to X \;\middle|\; \begin{array}{c} f \text{ has finite range} \\ \text{and is } \Sigma\text{-measurable} \end{array} \right\}.$
- Agent i has $\succsim_i$ over F represented by maximization of

$$\int_S u_i(f(s))dp_i$$

II.2 Definitions

- Pareto domination: $f \succ_P g$ iff for all $i \in N$, $f \succsim_i g$, and for some $k \in N$, $f \succ_k g$
- For a pair (f, g), agent i is *involved* in (f, g) if $u_i(f(\cdot)) \neq u_i(g(\cdot))$.
- $N(f, g) \subset N$ the agents who are involved in the pair (f, g).
- (f, g) is an *improvement* if $N(f, g) \neq \varnothing$ and, for all $i \in N(f, g)$, $f \succ_i g$.
- We also say that f *improves upon* g. Denoted $f \succ_* g$.

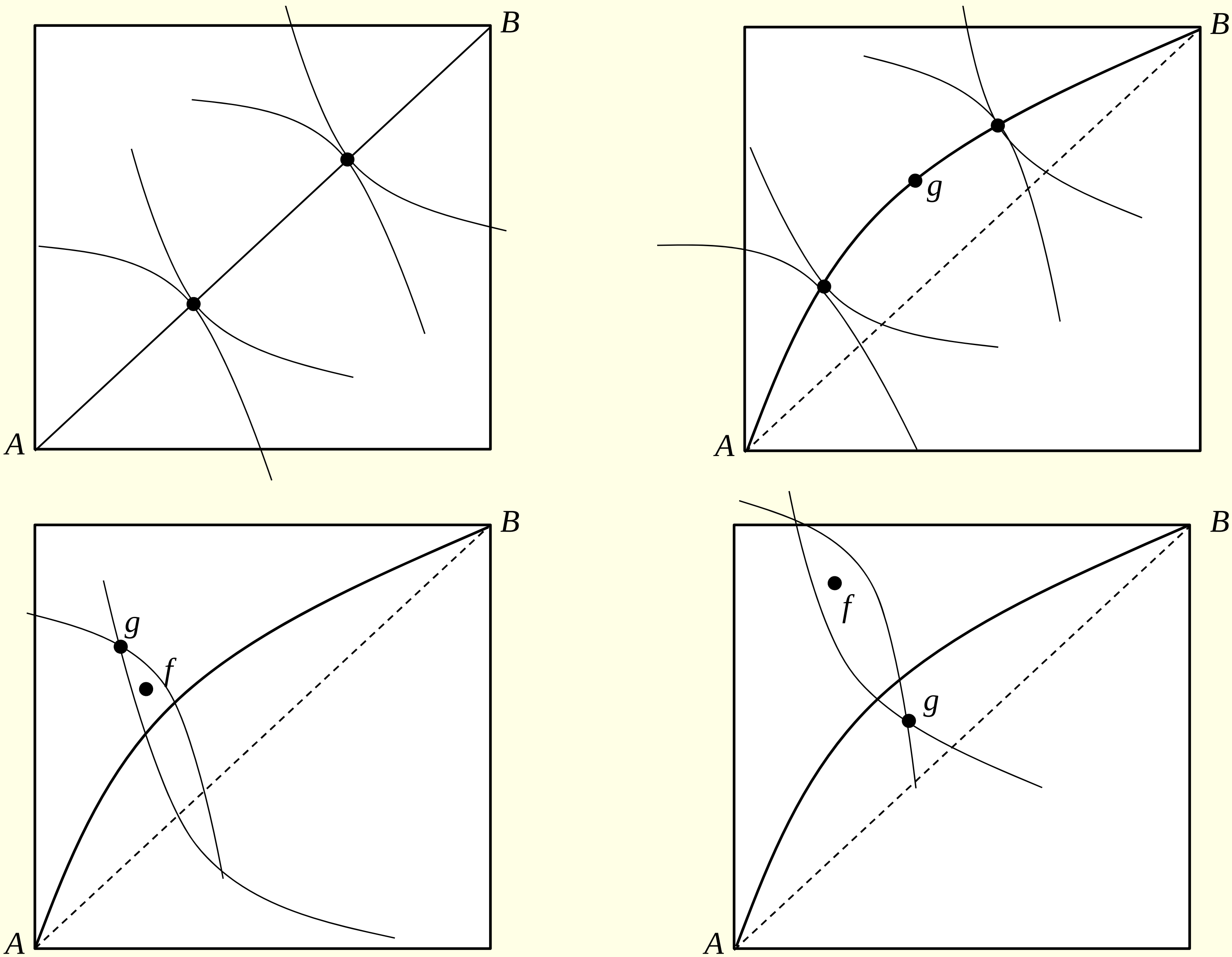
II.3 No-Betting-Pareto

Given $f, g \in F$, we say that f *No-Betting Pareto dominates* g, denoted $f \succ_{NBP} g$, if:

- f improves upon g;
- There exists a probability measure p_0 such that, for all $i \in N(f, g)$,

$$\int_S u_i(f(s))dp_0 > \int_S u_i(g(s))dp_0.$$

- Notice that we do not require that the agents agree on the distributions of f, g, only that they can rationalize trade by hypothetical beliefs.



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III. Implications

III.1 Characterization

Proposition 1 *Consider acts f and g with $N(f,g) \neq \varnothing$. There exists a probability vector p_0 such that, for all $i \in N(f,g)$,*

$$\int_S u_i(f(s))dp_0 > \int_S u_i(g(s))dp_0$$

if and only if, for every distribution over the set of agents involved, denoted by $\lambda \in \Delta(N(f,g))$, there exists a state $s \in S$, such that

$$\sum_{i \in N(f,g)} \lambda(i) u_i(f(s)) > \sum_{i \in N(f,g)} \lambda(i) u_i(g(s)).$$

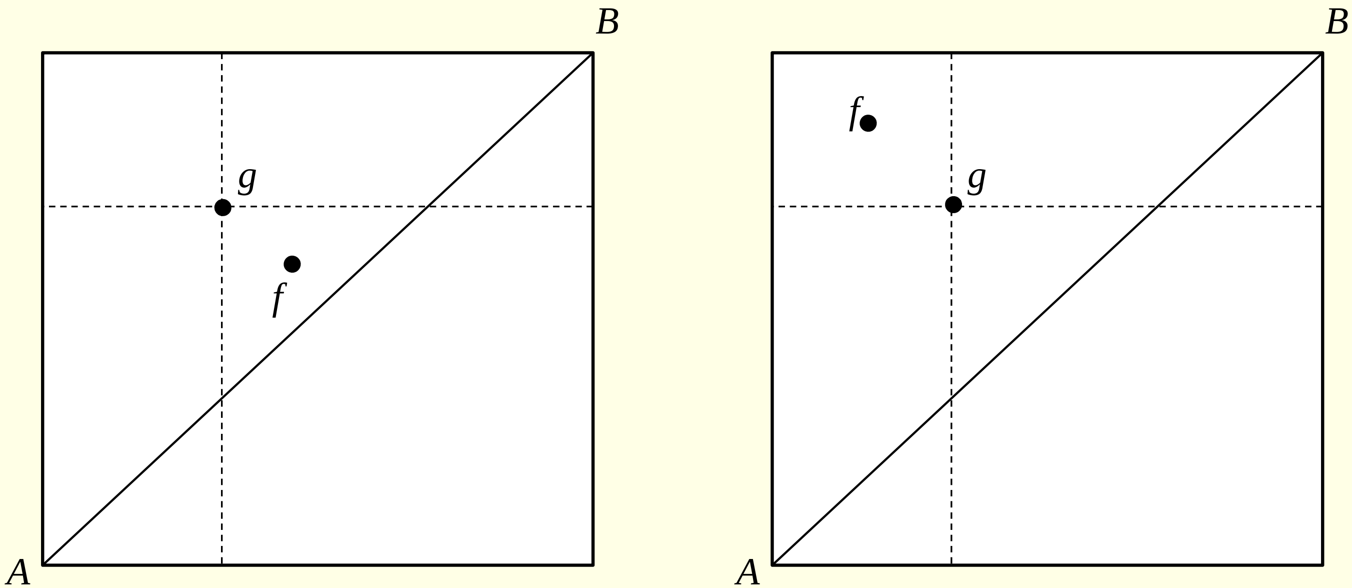
– The λ-average can be interpreted as a utilitarian social welfare function, or as an expected utility calculation behind the veil of ignorance.

III.2 Speculative Trade

- $X = \mathbb{R}^N$, $x = (x_1, ..., x_n) \in X$, x_i – wealth.
- $u_i((x_1, ..., x_n)) = u_i(x_i)$
- u_i is differentiable, strictly monotone and (weakly) concave.
- A pair (f, g) is *feasible* if $\sum_{i \in N(f,g)} f(s)_i \leq \sum_{i \in N(f,g)} g(s)_i$ for all s.
- A feasible improvement (f, g) is a *bet* if $g(s)_i$ is independent of s for each $i \in N(f, g)$.

III.3 No Betting

Proposition 2 *If* (f, g) *is a bet, then it cannot be the case that* $f \succ_{NBP} g$.



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III.4 What Else?

Let there be given utilities $(u_i)_i$ and two alternatives $f, g \in F$. The following are equivalent:

Theorem 1 *(i) There does* not *exist a probability vector* p_0 *such that, for all* $i \in N(f,g)$,

$$\int_S u_i(f(s))dp_0 > \int_S u_i(g(s))dp_0$$

(ii) There exists an alternative $d \in F$ *satisfying*

$$\sum_{i \in N} d(s)_i = 0 \qquad \forall s \in S.$$

that also has the following property: for every $g' \in F$ *such that* $g'(s)_i$ *is independent of* s *for each* $i \in N(f,g)$ *and lies in the interior of* R*, and for every beliefs* $(p_i)_i$ *such that* $f \succ_* g$*, there exists* $\alpha > 0$ *such that* $(g' + \alpha d, g')$ *is a bet (for the utilities* $(u_i)_i$ *and the beliefs* $(p_i)_i$*).*

- The bet can be constructed without knowing the actual p_i's.

III.5 Transitivity

Proposition 3 *The relation $\succ_{NBP}$ is acyclic but it need not be transitive.*

For example, let $p_1 = (1, 0)$, $p_2 = (0, 1)$ and

		State s	State t
g :	Agent 1	0	0
	Agent 2	0	0

		State s	State t
h :	Agent 1	$+2$	-1
	Agent 2	-3	$+2$

		State s	State t
f :	Agent 1	$+4$	-4
	Agent 2	-4	$+4$

III.5 In Addition...

- Denote the transitive closure of $\succ_{NBP}$ by $\succ_{NBP}^{t}$.
- The range of u is *rectangular* if for every $(f_i)_i \in F^n$, there exists $f^* \in F$ such that, for all $i \in N$, $u_i(f^*(s)) = u_i(f_i(s))$ for all $s \in S$.

Proposition 4 *Assume that $range(u)$ is rectangular and convex. Then* $\succ_{NBP}^{t} = \succ_{*}$.

- However, this ignores feasibility constraints.
- Define $f \succ_{fNBP} g$ if (f, g) is feasible and $f \succ_{NBP} g$, then
- If (f, g) is a bet, then it cannot be the case that $f \succ_{fNBP}^{t} g$.

IV. Ambiguity

IV.1 The Setting

- Suppose that each agent has a set of priors and maximizes the minimum (over this set of priors) expected utility.

- We similarly adjust our no-betting Pareto criterion, saying that $f \succ_{NBP} g$ if f is an improvement of g and there is a single set of priors with respect to which every involved agent finds f better than g.

- Suppose f is an improvement of g. Then any trade that can be justified with a single prior p_0 can be justified with a set of priors, namely $\{p_0\}$.

- Does the set-valued approach allows us to justify anything else?

IV.2 Betting

- Let (f, g) be a bet. Then it cannot be that $f \succ_{NBP} g$ under ambiguity.

- Because (f, g) is a bet, we know that there any single probability, there is some agent who prefers g to f. But then given any set of probabilities, there must always be an agent who prefers g to f when the latter is evaluated according to the worst probability.

IV.3 Other Trades

There may exist (f, g) for which it is impossible that $f \succ_{NBP} g$ in the absence of ambiguity, but $f \succ_{NBP} g$ under ambiguity.

For example:

	s_1	s_2
f	.4, .4	.4, .4
g	1, 0	0, 1

Then we can have $f \succ_{NBP} f$ under ambiguity, but not risk. This appears to open up a prospect for risk-sharing that we would like to allow.

Another example:

	s_1	s_2	s_3	s_4
f	$-2, 1$	$1, -2$	$0, -1$	$-1, 0$
g	$1, -1$	$-1, 1$	$0, -1$	$-1, 0.$

Again, we can have $f \succ_{NBP} f$ under ambiguity, but not risk. Here, however, in moving from g to f, the agents have essentially reversed the roles of states s_1 and s_2, while paying one unit in the process.

VI. Discussion

- Return to Cathy and Doug, whose differing beliefs created gains from trade.
- Why should we disapprove of their willingness to trade? This seems to require more than a revealed preference argument.
- The argument appears to be that at least one of them has to have "incorrect" beliefs. But if we have a pure calculation-of-future-utility view, it should suffice to explain to them that their beliefs cannot be correct. If they are not amenable to such explanation, then it is unclear how we should interpret their probabilities.
- If there is something more to utility than just calculation of future utilities, should we reject their preferences?