Psychophysical Foundations of the Cobb-Douglas Utility Function*

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Abstract

Relying on a literal interpretation of Weber's law in psychophysics, we show that a simple condition of independence across good categories implies the Cobb-Douglas preferences.

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1 Introduction

Cobb-Douglas utility functions enjoy great popularity as an example of consumer preferences. They exhibit many nice theoretical properties (such as monotonicity, convexity, homotheticity) as well as analytic tractability, and these make them favorites for classroom and estimation alike. Clearly, these functions are limited in many ways, and even an intermediate micro course would soon move on to examples of preferences with richer substitution and complementarity effects. Still, Cobb-Douglas preferences seem to retain their primacy in textbooks and in empirical work despite – and most probably because – of their simplicity.

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This note points out that there are other reasons for which one might be interested in Cobb-Douglas preferences. Studies in psychophysics suggest that humans do not have a perfect perception ability, and, further, that perception behaves in a rather orderly way. Specifically, Weber's Law (Weber, 1834) offers the logarithmic function as the scaling of a stimulus such that the "just noticeable difference" becomes constant. A result in Argenziano and Gilboa (2017) shows that a mild separability condition (across goods, or good categories) implies that preferences over bundles are representable by linear functions of the logarithms. Combined, one obtains the Cobb-Douglas preferences (in their logarithmic representation). While these preferences remain over-simplistic for many purposes, it is interesting to know that, among the classes of simple functions, there is a particular reason to choose them, beyond mathematical convenience.

2 Weber's Law and Semi-Orders

Weber (1834) was interested in the minimal degree of change in a stimulus needed for this change to be noticed. For a physical stimulus (such as weight or length) of size S, let ΔS be the minimal increase of the stimulus level so that $(S + \Delta S)$ can be discerned as larger than S at least 75% of the trials. *Weber's law* states that this threshold behaves proportionately to S. That is, there exists a constant $c > 1$ that

$$(S + \Delta S)/S = c.$$

Thus, if the base-level stimulus is multiplied by a factor $a > 0$, the minimal change required to be noticed (with the same threshold probability) is $a\Delta S$. Equivalently, a change ΔS will be noticed only if

$$\log (S + \Delta S) - \log (S) > \delta \equiv \log (c) > 0. \tag{1}$$

This law is considered a rather good first approximation and it appears in most introductory psychology textbooks.[1]

Luce (1956) used this observation to refine the model of consumer choice. In a famous example, he argued that one cannot claim to have strict preferences between a cup of coffee with n and one with $(n+1)$ grains of sugar, for any n. Hence, two such cups would be equivalent in the eyes of the decision maker. This implies that we are bound to observe violations of transitivity of preferences: we will often observe a long chain of equivalences between close quantities, while the alternatives at the ends of the chain are not indifferent.

Luce therefore defined binary relations that he dubbed *semi-orders*, allowing for some types of intransitive indifferences. For the sake of our discussion, we can think of a semi-order as a binary relation $\succ$, denoting strict preference, that can be represented by a pair (u, δ) where u is a utility function on the set of alternatives and $\delta > 0$ is a threshold – called the *just noticeable difference* (jnd) – such that, for every x, y,

$$x \succ y \qquad iff \qquad u(x) - u(y) > \delta \tag{2}$$

In the absence of (strict) preference between two alternatives, x, y, that is, if neither $x \succ y$ nor $y \succ x$ holds, we will write $x \smile y$. If $\succ$ is a semi-order, it follows that $\smile$ is a reflexive and symmetric relation, and, indeed, for every x, y,[2]

$$x \smile y \qquad iff \qquad |u(x) - u(y)| \leq \delta \tag{3}$$

Given a semi-order $\succ$, one can also define the associated equivalence

[1] It is often mentioned in the context of the Weber-Fechner law. Fechner (1860) was interested also in subjective perception. Over the past decades, Stevens's power law is considered to be a better approximation of subjective perceptions than is Fechner's law. However, as far as discernibility is concerned, Weber's law probably still holds the claim to be the best first approximation. See Algom (2001).

[2] One can also think of semi-orders where strict preference is represented by a weak inequality, and indifference $\smile$ – by a strict inequality. See Beja and Gilboa (1992) for details and necessary and sufficient conditions for the existence of each representation.

relation, $\sim$, as follows: for every x, y, $x \sim y$ if and only if

$$\begin{array}{lrcl} \forall z, & x & \succ & z \Leftrightarrow y \succ z \\ & & & and \\ \forall z, & z & \succ & x \Leftrightarrow z \succ y \end{array}$$

Naturally, $x \sim y$ implies $x \smile y$, but the converse is not generally true. Indeed, $\sim$ is an equivalence relation, and, given a representation of $\succ$, (u, δ), one may assume that it also satisfies

$$x \sim y \qquad iff \qquad u(x) = u(y) \tag{4}$$

Under some richness conditions, this will follow from (2).

It is easy to see that the utility function u in (2) is not only ordinal. One can use a monotone transformation of u, $f : \mathbb{R} \to \mathbb{R}$, to represent preferences by $v = f(u)$, only if, for every $\alpha, \beta \in \mathbb{R}$,

$$|\alpha - \beta| \leq \delta \qquad iff \qquad |f(\alpha) - f(\beta)| \leq \delta$$

Thus, the function f above can be any arbitrary strictly increasing function over the $[0, \delta]$ interval, as long as $f(\delta) - f(0) = \delta$, but the number of "$\delta$-steps" between two alternatives has to be respected by any function that represents preferences, whether measured on the original u scale or on the transformed v scale. Accordingly, the number of just-noticeable-difference (δ) steps between alternatives can provide a measure of the intensity of preferences and thereby to provide empirical meaning to claims such as "the marginal utility of money is decreasing".

3 Aggregation of Semi-Orders

We cite a result regarding the aggregation of n semi-orders, each defined on $\mathbb{R}_+$, to a semi-order defined on their product space, $\mathbb{R}_+^n$. The result is a special case of the main result of Argenziano and Gilboa (2017), cited here for completeness of exposition.[3]

There are n product categories. Let $x_i > 0$ denote an amount of product category $i \leq n$, so that consumption bundles are vectors

$$x = (x_1, ..., x_n) \in X \equiv \mathbb{R}_+^n.$$

For each category $i \leq n$ the consumer has semi-ordered preferences $\succ_i$ on $\mathbb{R}_+$ that are represented by (v_i, δ_i) as follows: for every $x_i, y_i > 0$

$$\begin{array}{lclcl} x_i & \succ & _i\, y_i & iff & v_i(x_i) - v_i(y_i) > \delta_i \\ x_i & \smile & _i\, y_i & iff & |v_i(x_i) - v_i(y_i)| \leq \delta_i \end{array} \tag{5}$$

Preferences over each category are assumed to be monotonically increasing, and the main information conveyed by v_i is the number of jnd's that one can find between two values x_i and y_i.

We assume that v_i is strictly monotone and continuous, and that $\delta_i > 0$. We will also assume that for each i, $\succ_i$ is *unbounded from above*: for every $x_i \in \mathbb{R}_+$, there exists $y_i \in \mathbb{R}_+$ such that $y_i \succ_i x_i$. The representation (5) implies that v_i is unbounded, and its continuity implies that its range is $R_i \equiv image\,(v_i) = [v_i\,(0)\,, \infty)$.

We assume that the consumer has semi-ordered preferences $\succ$ on the set of bundles $\mathbb{R}_+^n$ that is represented by (u, δ_0) with $\delta_0 > 0$. Without loss of generality we assume that $\delta_0 = 1$. Thus, $u : \mathbb{R}_+^n \to \mathbb{R}$ is such that, for every

[3] The result in Argenziano and Gilboa (2017) is stated for vectors in $\mathbb{R}^m$ for each of n individuals. Here we consider but one individual, and each component is a good, or a goods category. Mathematically, we cite the result for the special case of $m = 1$.

$x, y \in \mathbb{R}^n_+$,

$$\begin{array}{ccccc} x & \succ & y & iff & u(x) - u(y) > 1 \\ x & \smile & y & iff & |u(x) - u(y)| \leq 1 \end{array} \tag{6}$$

We similarly assume that u is continuous.

For $z \in X$ and $x_i > 0$ we denote by $(z_{-i}, x_i) \in X$ the bundle obtained by replacing the i-th component of z, z_i, by x_i. The main assumption we use is[4]

Separability: For every i, every $z \in X$ and every $x_i, y_i > 0$,

$$(z_{-i}, x_i) \succ (z_{-i}, y_i) \qquad iff \qquad x_i \succ_i y_i$$

Observe that, if all jnd's were zero, Separability would boil down to simple monotonicity. In the presence of semi-ordered preferences, Separability still states that, if we focus on category i, and hold all other categories fixed, the consumer's ability to discern differences in quantities is independent of the quantities of the other product categories. This assumption may evidently be violated, especially if there are complementarity and substitution effects between the categories. But it seems to be a reasonable benchmark.

For the statement of the result we need the following definition: a *jnd-grid* of allocations is a collection $A \subset X$ such that, for every $x, y \in A$ and every $i \in N$,

$$v_i(x_i) - v_i(y_i) = k_i \delta_i \qquad \text{for some } k_i \in \mathbb{Z}$$

Thus, a jnd-grid is a countable subset of bundles, such that the utility differences between any two elements thereof, for any category, is an integer multiple of that category's jnd.

We can now cite

[4] In Argenziano-Gilboa (2017) the corresponding assumption is referred to as "Consistency". While Consistency was suggested as a normative principle in the context of social choice, here it is but a descriptive assumption on individual preferences.

Theorem 1 *(Argenziano-Gilboa, 2017) Let there be given* $(\succ_i)_{i \leq n}$, $((v_i, \delta_i))_{i \leq n}$, $\succ$ *and* u *as above. Separability holds iff there exists a strictly monotone, continuous*

$$g : \prod_{i=1}^{n} R_i \to \mathbb{R}$$

such that for every $x \in X$

$$u(x) = g(v_1(x_1), ..., v_n(x_n))$$

and, for every jnd-grid $A \subset X$ *there exists* $c \in \mathbb{R}$ *such that, for every* $x \in A$,

$$u(x) = c + \sum_{i=1}^{n} \frac{1}{\delta_i} v_i(x_i)$$

4 Cobb-Douglas Preferences

We now wrap up the above to conclude that Weber's law, interpreted literally, and coupled with the Separability assumption, yields Cobb-Douglas preferences. Indeed, let us assume that, for each category i, given any current quantity $x_i > 0$, the consumer would notice the difference Δx_i iff

$$\frac{x_i + \Delta x_i}{x_i} > c$$

for a fixed $c > 1$. Thus, $(x_i + \Delta x_i) \succ_i x_i$ iff

$$\log(x_i + \Delta x_i) - \log(x_i) > \delta_i \tag{7}$$

with $\delta_i > 0$. Thus, Weber's law applied to each good category i implies that the consumer has semi-ordered preferences over each category, which can be represented by the pair $(\log(x_i), \delta_i)$.

Further, assume that Separability holds. Then Theorem (1) implies that, on any jnd-grid,

$$u(x) = \sum_{i=1}^{n} \frac{1}{\delta_i} \log(x_i) \tag{8}$$

which are Cobb-Douglas preferences for coefficients $\alpha_i = \frac{1}{\delta_i}$. Note that, as mentioned in Section 2 in the presence of semi-ordered preferences, the consumer's utility function is not ordinal. In particular, if the function (8) is replaced by

$$w(x) = \prod_{i=1}^{n} (x_i)^{\alpha_i} \tag{9}$$

we will not obtain a representation of preferences as in (6).[5]

References

[1] Algom, D. (2001), "Psychophysics", in the *Encyclopedia of Cognitive Science*, Nature Publishing Group (Macmillan), London. 800-805.

[2] Argenziano, R. and I. Gilboa (2017), "Foundations of Weighted Utilitarianism", mimeo.

[3] Beja, A. and I. Gilboa (1992), "Numerical Representations of Imperfectly Ordered Preferences (A Unified Geometric Exposition)", *Journal of Mathematical Psychology*, **36**: 426-449.

[4] Fechner, G. T. (1860), *Elemente der Psychophysik*, 2 vol. (Elements of Psychophysics).

[5] Luce, R. D. (1956), "Semiorders and a Theory of Utility Discrimination", *Econometrica,* **24**: 178-191.

[6] Weber, E. H. (1834), *De Tactu* ("Concerning Touch").

[5] In this case, the function w would represent preferences in multiplicative way, that is, for every x, y,

$$x \succ y \qquad iff \qquad w(x)/w(y) > c$$

for $c > 1$.