JOURNAL OF MATHEMATICAL PSYCHOLOGY **34**, 371–392 (1990)

A Necessary but Insufficient Condition for the Stochastic Binary Choice Problem

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The "stochastic binary choice problem" is the following: Let there be given n alternatives, to be denoted by $N=\{1, ..., n\}$. For each of the $n!$ possible linear orderings $\{\succ^m\}_{m=1}^{n}$ of the alternatives, define a matrix $Y_{n\times n}^{(m)} (1 \leqslant m \leqslant n!)$ as follows:

$$Y_{ab}^{(m)} = \begin{cases} 1 & a \succ^m b \\ 0 & \text{otherwise.} \end{cases}$$

Given a real matrix $Q_{n\times n}$, when is Q in the convex hull of $\{Y^{(m)}\}_m$?

In this paper some necessary conditions on Q—the "diagonal inequality"—are formulated and they are proved to generalize the Cohen–Falmagne conditions. A counterexample shows that the diagonal inequality is insufficient (as are hence, perforce, the Cohen–Falmagne conditions). The same example is used to show that Fishburn's conditions are also insufficient.

1. BACKGROUND AND MOTIVATION

The stochastic binary choice problem arises in the context of inconsistent decision. Suppose we observe an individual choosing between pairs of alternatives under seemingly unchanged circumstances, who fails to stick to a single alternative out of each pair. We may disregard this individual, dubbing him "irrational," but the unfortunate prevalence of the phenomenon calls for a second thought: it may well be that the decision maker under discussion is completely rational, but some of the relevant variables which affect his/her decisions are not known to us, and consequently the circumstances which seem to be the same are in fact quite different.

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An earlier but almost identical version of this paper was circulated as Working Paper No. 19-85 of the Foerder Institute for Economic Research, Tel Aviv University. The results presented here were also included in my master's thesis written under the supervision of Professor David Schmeidler at the Department of Economics, Tel Aviv University. I am grateful to Professor Schmeidler for his guidance. I would also like to thank Professors Salvador Barbera, Peter Fishburn, David Hartvigsen, A. A. J. Marley, Dov Monderer, Hugo Sonnenschein, and Eitan Zemel for discussions, references, and encouragment.

0022-2496/90 $3.00

Since we may assume that the probabilities Q_{ab} (of preferring alternative a to b) are observable, the question is: What are the conditions on these probabilities to justify the above explanation for consistency? Or, equivalently, when can we say for sure that, no matter what relevant aspects of the decision we have failed to observe, the individual whose behavior is represented by the matrix Q is irrational?

Another interpretation of this problem is the following: let there by given a population, distributed among the $n!$ possible preference orders according to some probability vector $p=(p_1, ..., p_{n!})$. The matrix $Q=\sum_{m=1}^{n!} p_m Y^{(m)}$ is the pairwise majority vote of this population. The question is, therefore: What are the matrices Q that can be the majority vote of some population?

Another problem, closely related to the one discussed here, is the following: for each $>^m$ define a vector $Z^{(m)}=(Z_{a,A}^{(m)})_{a\in A\subseteq N}$ (for every subset A of N and every element a of A there is an entry in the vector $Z^{(m)}$) by

$$Z_{a,A}^{(m)}=\begin{cases}1 & (\forall b\in A, b\neq a)(a>^m b)\\ 0 & \text{otherwise.}\end{cases}$$

Given a real vector $R=(R_{a,A})_{a\in A\subseteq N}$, when is it a convex combination of $\{Z^{(m)}\}_{m=1}^{n!}$?

The interpretation of this problem is, of course, very similar, except that we assume that the probabilities $R_{a,A}$ are given for every $A\subseteq N$, while the previous problem assumed these data to be given only for $|A|=2$.

Necessary conditions on the vector R (to be in the convex hull of $\{Z^{(m)}\}_m$) were formulated by Block and Marschak (1960), and their sufficiency was provided by Falmagne (1978). Block and Marschak have also formulated necessary conditions for the stochastic order problem discussed in this paper, but they have not proved them to be sufficient. McFadden and Richter (1970) provided a counterexample which showed that the sufficiency conjecture was false. (This example was also found independently by Cohen and Falmagne (1978), Dridi (1980), Souza (1983), and Fishburn (1988).) Cohen and Falmagne (1978) and Fishburn (1988) also suggested new sets of necessary conditions, without solving the question of their sufficiency which will be solved in the sequel. Other works on this problem are McLennan (1984), Barbera and Pattanaik (1986), and Barbera (1985). Surveys which also contain additional references are given by Fishburn and Falmagne (1988) and Marley (1989).

In the following subsection we cite both Block and Marschak's necessary conditions (called "the triangle inequality") and the proof of their insufficiency. Section 2 will formulate and prove the necessity of stricter conditions, to be named "the diagonal inequality," Section 3 is devoted to the proof of the insufficiency of the diagonal inequality, while Section 4 includes some remarks concerning this paper's results in relation to the literature. More specifically, it proves that the Cohen–Falmagne condition is a special case of the diagonal inequality (hence also insufficient) and that Fishburn's conditions are insufficient (even in conjunction with the diagonal inequality).

1.1. *The Triangle Inequality*

We begin with some trivial conditions that any matrix $Q \in \operatorname{conv}\{Y^{(m)}\}_m$ must satisfy (where "conv" means convex hull):

$$\text{(i)} \quad Q_{ab} \geqslant 0 \qquad \forall a, b \in N$$

$$\text{(ii)} \quad Q_{aa} = 0 \qquad \forall a \in N$$

$$\text{(iii)} \quad Q_{ab} + Q_{ba} = 1 \qquad \forall a \neq b \in N.$$

Next we turn to the triangle inequality. It is easily seen that, since $\succ^m$ is transitive for all $m \leqslant n!$, each $Y^{(m)}$ has to satisfy

$$Y_{ab}^{(m)} + Y_{bc}^{(m)} \leqslant 1 + Y_{ac}^{(m)}, \qquad \forall a, b, c \in N.$$

Hence, a convex combination of $\{Y^{(m)}\}_m$ will also satisfy this condition. Using condition (iii) one may conclude that, for all $Q \in \operatorname{conv}\{Y^{(m)}\}_m$,

$$Q_{ab} + Q_{bc} \geqslant Q_{ac}, \qquad \forall a, b, c \in N.$$

This condition is the famous triangle inequality.

It should be noted that a necessary and sufficient condition for Q to belong to $\operatorname{conv}\{Y^{(m)}\}_m$ must be representable in the form of finitely many linear inequalities (see Weyl (1935).) Hence it was natural to suspect that the triangle inequality was sufficient. However, the counterexample, which keeps being rediscovered, is the following: consider the matrix Q shown in Fig. 1. Q satisfies the triangle inequality. Assume that Q is indeed in the convex hull of $\{Y^{(m)}\}_m$. Now, if $p_m > 0$, $Y^{(m)}$ must be zero where Q is zero. The preference orders $\{\succ^m\}_m$, the corresponding matrices of which satisfy this requirement, must satisfy

$$\begin{array}{lll} 1 \succ^m 4 & 2 \succ^m 4 & 3 \succ^m 5 \\ 1 \succ^m 5 & 2 \succ^m 6 & 3 \succ^m 6. \end{array} \qquad (*)$$

$$Q = \begin{vmatrix} 0 & 1/2 & 1/2 & 1 & 1 & 1/2 \\ 1/2 & 0 & 1/2 & 1 & 1/2 & 1 \\ 1/2 & 1/2 & 0 & 1/2 & 1 & 1 \\ 0 & 0 & 1/2 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1/2 & 0 \end{vmatrix}$$

FIGURE 1

Now consider the set of indices A, such that for every $m \in A$, (*) holds and $6 \succ^m 1$. A contains the four indices of the preference relations satisfying

$$2, 3 \succ^m 6 \succ^m 1 \succ^m 4, 5.$$

Similarly, let B be those indices m, the preference relations of which satisfy (*) and $5 \succ^m 2$. They are the four relations for which

$$1, 3 \succ^m 5 \succ^m 2 \succ^m 4, 6.$$

And finally denote by C the indices for which both (*) and $4 \succ^m 3$ hold. These preference relations satisfy

$$1, 2 \succ^m 4 \succ^m 3 \succ^m 5, 6.$$

It is easily seen that these three quadruples are pairwise disjoint. However, as $Q_{61} = Q_{52} = Q_{43} = \frac{1}{2}$,

$$\sum_{m \in A} p_m = \sum_{m \in B} p_m = \sum_{m \in C} p_m = \frac{1}{2}$$

has to hold, which is an obvious contradiction.

2. The Diagonal Inequality

Let there be given two sets of indices, A, $B \subseteq N$, such that $|A| = |B| = k$ $(1 \leqslant k \leqslant n)$. Consider the submatrix of dimension $k \times k$, corresponding to $A \times B \subseteq N \times N$. ($A$ and B are not necessarily disjoint.) Enumerate the elements of A and B in an arbitrary way, $A = \{a_i\}_{i=1}^k$, $B = \{b_j\}_{j=1}^k$, and consider the diagonal $\{(a_i, b_i)\}_{i=1}^k$.

Now choose any matrix $Y^{(m)}$, corresponding to $\succ^m$, and consider its submatrix defined by A and B. Suppose that for $1 \leqslant i \neq j \leqslant k$ it is true that $Y^{(m)}_{a_i b_i} = Y^{(m)}_{a_j b_j} = 1$.

This implies $a_i \succ^m b_i$ and $a_j \succ^m b_j$. Surely, either $a_i \succ^m b_j$ or $a_j \succ^m b_i$ (or both) must hold. (Otherwise, $a_j \succ^m b_j \succsim^m a_i \succ^m b_i \succsim^m a_j$.)

Hence $Y^{(m)}_{a_i b_j} = 1$ or $Y^{(m)}_{a_j b_i} = 1$ (or both); that is, for every pair of 1's on the diagonal there must be at least one 1 off the diagonal.

As each $Y^{(m)}$ consists solely of zeros and ones, the number of 1's on the diagonal is

$$D = \sum_{i=1}^{k} Y^{(m)}_{a_i b_i}$$

and the number of 1's off the diagonal is

$$S = \sum_{1 \leqslant i \neq j \leqslant k} Y^{(m)}_{a_i b_j}.$$

Hence, each $Y^{(m)}$ satisfies

$$S \geqslant \binom{D}{2} = \frac{1}{2} D(D-1).$$

Let us now consider the plane DS, and translate the quadratic inequality into linear inequalities: for every r, $1 \leqslant r \leqslant k-1$ we draw the string connecting the two adjacent integer points on the parabola

$$\left(r, \binom{r}{2}\right) \quad \text{and} \quad \left(r+1, \binom{r+1}{2}\right).$$

Because D and S may assume only integer values for each $Y^{(m)}$, S must be above each of these strings. (For the integer points, this condition is equivalent to the quadratic one.)

The equation of the line connecting $(r, \binom{r}{2})$ and $(r+1, \binom{r+1}{2})$ is $S = r \cdot D - \binom{r+1}{2}$. This proves

THEOREM. *A necessary condition for a given matrix Q to belong to the convex hull of $\{Y^{(m)}\}_m$ is*

> *for every $k \leqslant n$, every $\{a_i\}_{i=1}^{k} \subseteq N$, every $\{b_j\}_{j=1}^{k} \subseteq N$, and every $1 \leqslant r \leqslant k-1$,*

$$\sum_{1 \leqslant i \neq j \leqslant k} Q_{a_i b_j} \geqslant \sum_{i=1}^{k} Q_{a_i b_i} - \frac{1}{2} r(r+1),$$

Remark 1. Choosing $k=2$, $A=\{a, b\}$, $B=\{b, c\}$ and $r=1$, one gets the necessary condtion

$$Q_{ac} + Q_{bb} \geqslant Q_{ab} + Q_{bc} - 1$$

or

$$Q_{ab} + Q_{bc} \leqslant 1 + Q_{ac}.$$

So the triangle inequality is a special case of the diagonal inequality.

Remark 2. The matrix Q of the famous example quoted above does not satisfy the diagonal inequality: Let $k=3$, $A=\{4, 5, 6\}$, $B=\{3, 2, 1\}$, for which $S=0$, $D=\frac{3}{2}$, and the condition

$$S \geqslant r \cdot D - \tfrac{1}{2} r(r+1)$$

does not hold for $r=1$.

3. The Insufficiency of the Diagonal Inequality

In this section we define the term "graph decomposition," then prove the existence of a graph that is not $\frac{2}{3}$-decomposable, and only afterwards prove the insufficiency of the diagonal inequality using the graph which is not $\frac{2}{3}$-decomposable.

3.1. *Definition of Graph Decomposition*

First we define the tournament graphs over N: a directed graph $G(N, E)$ is called *tournament* (over N) iff for any $a \neq b \in N$ either $(a, b) \in E$ or $(b, a) \in E$ (but not both), and for every $a \in N$, $(a, a) \notin E$. (See Roberts (1976, pp. 81–93) for definition and basic properties.)

The set of tournaments will be denoted by $\mathscr{E}$. (This notation as well as the rest of the discussion presupposes a given N. As long as no confusion can result, we will suppress unnecessary subscripts.)

Denote by $\mathscr{E}^T$ the transitive tournaments over N: $(G \in \mathscr{E})$ is transitive iff

$$(a, b), (b, c) \in E \Rightarrow (a, c) \in E,$$

$$\mathscr{E} \supset \mathscr{E}^T = \{G_m^T(N, E_m^T)\}_{m=1}^{n!}.$$

(The set of edges of G_m^T will henceforth be denoted by E_m^T.)

It is obvious that there is a one-to-one correspondence between $\{G_m^T\}_m$ and $\{Y^{(m)}\}_m$, since every transitive tournament defines a linear preference relation over N, and vice versa.

We are interested in probability distributions over $\mathscr{E}^T$. Let G_R^T be a random variable assuming values in $\mathscr{E}^T$ according to the probability vector $p = (p_1, ..., p_{n!})$. For $(a, b) \in N \times N$ define an event Pref_{ab} (a is preferred to b) as

$$\mathrm{Pref}_{ab} = \bigcup_{m \in M_{ab}} (G_R^T = G_m^T),$$

where $M_{ab} = \{1 \leqslant m \leqslant n! \mid (a, b) \in E_m^T\}$. (For $a = b$ $\mathrm{Pref}_{ab} = \varnothing$.) By this definition, $\mathrm{Prob}(\mathrm{Pref}_{ab}) = \sum_{m \in M_{ab}} p_m$.

Definition. A tournament $G(N, E) \in \mathscr{E}$ is μ-decomposable for $\mu \in [0, 1]$ if there exists a probability vector $p = (p_1, ..., p_{n!})$ such that for all $(a, b) \in E$,

$$\mathrm{Prob}(\mathrm{Pref}_{ab}) \geqslant \mu.$$

For instance, every $G \in \mathscr{E}$ is $\frac{1}{2}$-decomposable, since $p_m = 1/n!$ defines $\mathrm{Prob}(\mathrm{Pref}_{ab}) = \frac{1}{2}$ for all $a \neq b$. We would like to know whether every $G \in \mathscr{E}$ is $\frac{2}{3}$-decomposable for all n.

The negative answer is given in the next subsection.

3.2. *The Existence of a Tournament That Is Not $\frac{2}{3}$-Decomposable*

We will need:

LEMMA. *Let $G \in \mathscr{E}$ be $\frac{2}{3}$-decomposable, and suppose that (a, b), (b, c), $(c, a) \in E$. Then, if p is a probability vector of a G-decomposition, and* Prob *denotes the probability measure defined by p,*

(i) $\operatorname{Prob}(\operatorname{Pref}_{ba} \cap \operatorname{Pref}_{cb}) = 0$

(ii) $\operatorname{Prob}(\operatorname{Pref}_{ab}) = \frac{2}{3}$.

Proof. (i) If $\operatorname{Prob}(\operatorname{Pref}_{ba} \cap \operatorname{Pref}_{cb}) = \varepsilon > 0$, then

$$\begin{aligned}\operatorname{Prob}(\operatorname{Pref}_{ba} \cup \operatorname{Pref}_{cb}) = {} & \operatorname{Prob}(\operatorname{Pref}_{ba}) + \operatorname{Prob}(\operatorname{Pref}_{cb}) \\ & - \operatorname{Prob}(\operatorname{Pref}_{ba} \cap \operatorname{Pref}_{cb}) \leqslant \tfrac{2}{3} - \varepsilon,\end{aligned}$$

whence $\operatorname{Prob}(\operatorname{Pref}_{ab} \cap \operatorname{Pref}_{bc}) \geqslant \frac{1}{3} + \varepsilon$. But, since for all $G^T \in \mathscr{E}^T$ in which (a, b), $(b, c) \in E^T$, it is true that $(a, c) \in E^T$,

$$\operatorname{Prob}(\operatorname{Pref}_{ac}) \geqslant \tfrac{1}{3} + \varepsilon$$

and $\operatorname{Prob}(\operatorname{Pref}_{ca}) < \frac{2}{3}$, in contradiction to the $\frac{2}{3}$-decomposability of G.

(ii) By definition of decomposability,

$$\operatorname{Prob}(\operatorname{Pref}_{ab}) \geqslant \tfrac{2}{3}.$$

If the inequality is strict, $\operatorname{Prob}(\operatorname{Pref}_{ba}) < \frac{1}{3}$. This implies

$$\operatorname{Prob}(\operatorname{Pref}_{ba} \cup \operatorname{Pref}_{cb} \cup \operatorname{Pref}_{ac}) \leqslant \operatorname{Prob}(\operatorname{Pref}_{ba}) + \operatorname{Prob}(\operatorname{Pref}_{cb}) + \operatorname{Prob}(\operatorname{Pref}_{ac}) < 1$$

whence

$$\operatorname{Prof}(\operatorname{Pref}_{ab} \cap \operatorname{Pref}_{bc} \cap \operatorname{Pref}_{ca}) > 0$$

which is impossible since all the graphs in $\mathscr{E}^T$ are transitive. ∎

The Definition of the Graph. We need 54 vertices:

- 6 vertices will be called a, b, c, d, e, f.
- 48 vertices will be called (i, j, k) for specific values of i, j, k:
  - — i will assume the values $\{1, 2, 3\}$.
  - — For each of these i-values, j will assume the values $\{1, ..., 8\}$.
  - — The possible values of k depend upon the value of j, according to the following table:

j-Value	1	2	3	4	5	6	7	8
k Values	1	1, 2	1, 2	1	1, 2, 3	1, 2, 3	1, 2, 3	1

(Thus, typical vertices are (1, 1, 1), (3, 6, 3), etc.) The edges between the vertices are the following:

$$(a, b), (b, c), (c, a) \in E; (d, e), (e, f), (f, d) \in E$$
$$\{(x, y) \mid x \in \{a, b, c\}, y \in \{d, e, f\}\} \subseteq E.$$

(*abc* is a circle and so is *def*, where all the edges between them are directed from *abc* to *def*; see also Fig. 2.) To define the edges of the vertices (i, j, k) we will need a few abbreviations.

First, we define only those edges touching the vertices $\{(1, j, k)\}_{j,k}$, where the edges touching $\{(2, j, k)\}_{j,k}$ and $\{(3, j, k)\}_{j,k}$ will be defined according to a cyclic symmetry: the edges of $\{(2, j, k)\}_{j,k}$ are like those of $\{(1, j, k)\}_{j,k}$, where d is replaced by e, e by f, and f by d. The edges of $\{(3, j, k)\}_{j,k}$ are again like those of $\{(1, j, k)\}_{j,k}$, where d is replaced by f, e by d, and f by e.

Next, we define some abbreviations:

(1) Five vertices $(x_1, x_2, x_3, x_4, x_5)$ are in an *A-structure* if (x_1, x_5), (x_5, x_2), (x_3, x_5), $(x_5, x_4) \in E$ (see Fig. 3).

(2) Six vertices $(x_1, x_2, x_3, x_4, x_5, x_6)$ are in an *upper B-structure* if

(i) (x_2, x_5), (x_5, x_3), $(x_4, x_5) \in E$

(ii) $(x_1, x_2, x_5, x_4, x_6)$ are in an A-structure. (see Fig. 4).

(3) Six vertices $(x_1, x_2, x_3, x_4, x_5, x_6)$ are in a *lower B-structure* if

(i) (x_5, x_1), (x_2, x_5), $(x_5, x_3) \in E$

(ii) $(x_1, x_5, x_3, x_4, x_6)$ are in an A-structure, (see Fig. 5).

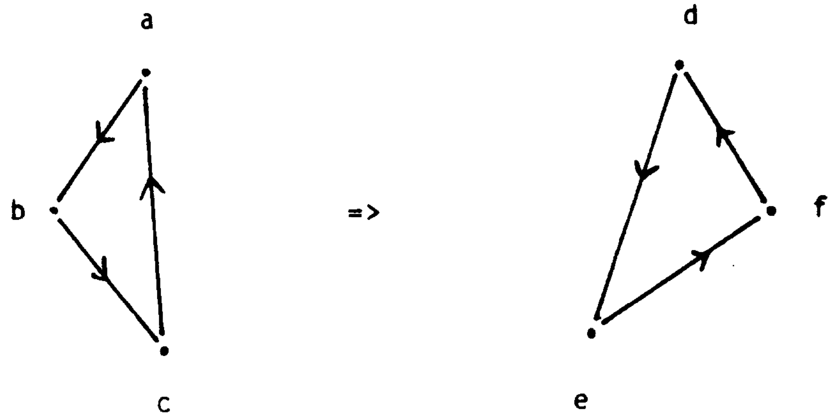
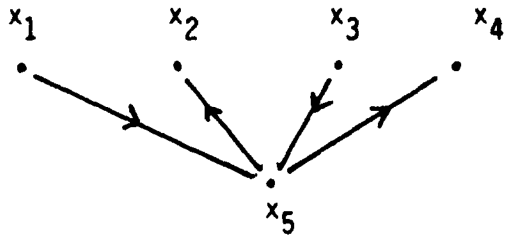


FIGURE 2



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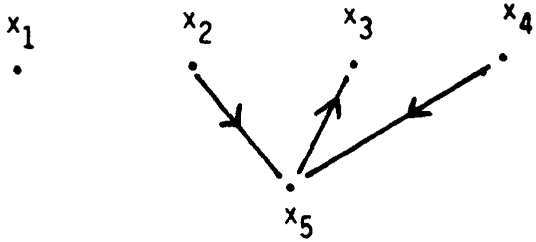

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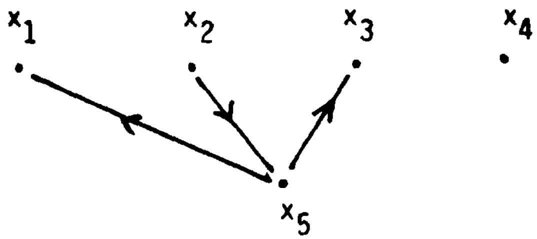

FIGURE 5

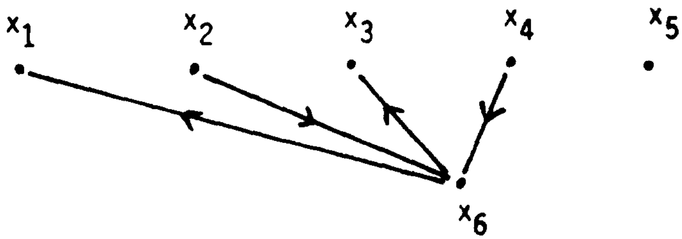

FIGURE 6

(4) Eight vertices $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ are in an *upper C-structure* if

(i) $(x_6, x_1), (x_2, x_6), (x_6, x_3), (x_4, x_6) \in E$

(ii) $(x_1, x_6, x_4, x_5, x_7, x_8)$ are in an upper B-structure (see Fig. 6).

(5) Eight vertices $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ are in a *lower C-structure* if

(i) $(x_6, x_2), (x_3, x_6), (x_6, x_4), (x_5, x_6) \in E$

(ii) $(x_1, x_2, x_6, x_5, x_7, x_8)$ are in a lower B-structure, (see Fig. 7).

With these abbreviations we finally specify the direction of edges touching the vertices $\{(1, j, k)\}_{j,k}$:

For $j=1$ $(a, c, d, f, (1, 1, 1))$ are in an A-structure.

For $j=2$ $(a, d, f, c, (1, 2, 1), (1, 2, 2))$ are in a lower B-structure.

For $j=3$ $(d, a, c, f, (1, 3, 1), (1, 3, 2))$ are in an upper B-structure.

For $j=4$ $(d, a, f, c, (1, 4, 1))$ are in an A-structure.

For $j=5$ $(a, b, d, c, f, (1, 5, 1), (1, 5, 2), (1, 5, 3))$ are in a lower C-structure.

For $j=6$ $(a, d, c, e, f, (1, 6, 1), (1, 6, 2), (1, 6, 3))$ are in an upper C-structure.

For $j=7$ $(a, d, e, b, c, (1, 7, 1), (1, 7, 2), (1, 7, 3))$ are in a lower C-structure.

For $j=8$ $(d, b, e, c, (1, 8, 1))$ are in an A-structure.

Since all the structures defined for $\{(1, j, k)\}_{j,k}$ do not involve edges touching the vectices $\{(i, j, k)\}_{j,k}$ for $i \neq 1$, the symmetric structures defined for $\{(2, j, k)\}_{j,k}$ and $\{(3, j, k)\}_{j,k}$ will have the same property, and hence these definitions do not contradict each other. The rest of the edges in G (that must belong to E for $G \in \mathscr{E}$ to hold) may be directed in an arbitrary way.

THE MAIN CLAIM. *The graph G defined above is not* $\frac{2}{3}$*-decomposable.*

Proof. Suppose G were $\frac{2}{3}$-decomposable, and let $p=(p_1, ..., p_{n!})$ be a decomposition probability vector. By the lemma proved above, $\text{Prob}(\text{Pref}_{ac}) = \frac{1}{3}$, whence there exists at least one index m for which $p_m > 0$ and $(a, c) \in E_m^T$. The lemma also

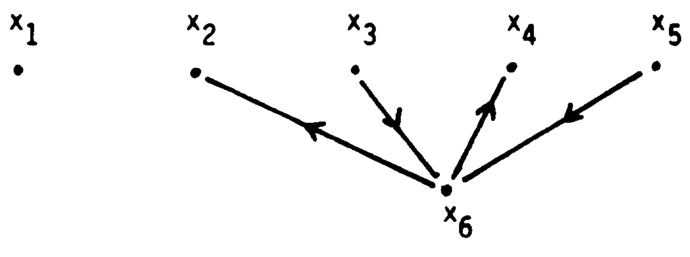

FIGURE 7

implies, as $(c, a) \in E$ has an opposite direction in G_m^T (i.e., $(a, c) \in E_m^T$ and not $(c, a) \in E_m^T$), that the two other edges in the same circle are directed in G_m^T as in G, that is, (a, b), $(b, c) \in E_m^T$, or $a \succ^m b \succ^m c$.

Similarly, the vertices d, e, f may appear in G_m^T in only one of the following three permutations:

$$d \succ^m e \succ^m f, \qquad e \succ^m f \succ^m d, \qquad f \succ^m d \succ^m e.$$

(The other three permutations are possible only if there are two edges in G, the direction of which is reversed in G_m^T, which is impossible by the lemma.)

CLAIM A. *The permutation of d, e, f in G_m^T cannot be $d \succ^m e \succ^m f$.*

Proof. For the proof we have to define some new abbreviations. These definitions are dependent upon both the original graph G and the new graph G_m^T discussed above:

(1) Four vertices (x_1, x_2, x_3, x_4) are in *position* 1 (see Fig. 8) iff:

(i) $(x_2, x_1), (x_4, x_3) \in E$

(ii) $(x_1, x_2), (x_3, x_4) \in E_m^T$

(iii) $(x_2, x_3) \in E_m^T$ or $(x_4, x_1) \in E_m^T$.

(In the figure, the straight line indicates the direction of the edges in G_m^T, where the arcs are original edges of G, which are reversed in G_m^T.)

(2) Four vertices (x_1, x_2, x_3, x_4) are in an *upper position* 2 (see Fig. 9) iff:

(i) $(x_2, x_1), (x_3, x_2) \in E$

(ii) $(x_1, x_2), (x_2, x_3), (x_3, x_4) \in E_m^T$.

(3) Four vertices (x_1, x_2, x_3, x_4) are in a *lower position* 2 (see Fig. 10) iff:

(i) $(x_3, x_2), (x_4, x_3) \in E$

(ii) $(x_1, x_2), (x_2, x_3), (x_3, x_4) \in E_m^T$.

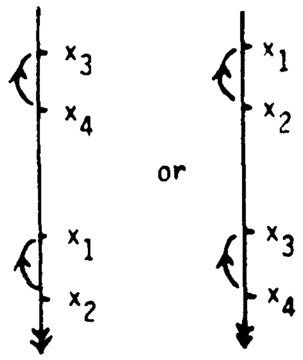

FIGURE 8

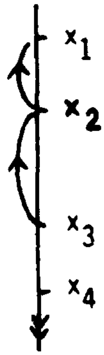

FIGURE 9

(4) Five vertices $(x_1, x_2, x_3, x_4, x_5)$ are in an *upper position* 3 (see Fig. 11) iff:

(i) $(x_3, x_2) \in E$

(ii) $(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5) \in E_m^T$.

(5) Five vertices $(x_1, x_2, x_3, x_4, x_5)$ are in a *lower position* 3 (see Fig. 12) iff:

(i) $(x_4, x_3) \in E$

(ii) $(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5) \in E_m^T$.

We have to prove a few auxiliary claims:

CLAIM A.1. *If there is a vertex x_5 such that $(x_1, x_2, x_3, x_4, x_5)$ are in an A-structure in G, it is false that (x_1, x_2, x_3, x_4) are in position* 1. (*Recall that the definitions of the structures refer to a single graph, which is always G in our discussion, whereas the definitions of the positions refer to both G and G_m^T.*)

Proof. By the definition of an A-structure,

$$(x_1, x_5), (x_5, x_2) \in E.$$

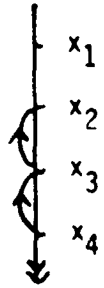

FIGURE 10

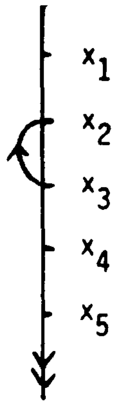

FIGURE 11

If (x_1, x_2, x_3, x_4) were in position 1, $(x_2, x_1) \in E$. But $(x_1, x_2) \in E_m^T$ (this edge is reversed in G_m^T), whence (x_1, x_5), $(x_5, x_2) \in E_m^T$. (The other two edges in the circle x_1, x_2, x_5 must have in G_m^T the same direction as in G.) Similarly, (x_3, x_5), $(x_5, x_4) \in E_m^T$.

By the definition of position 1, either $(x_2, x_3) \in E_m^T$ or $(x_4, x_1) \in E_m^T$. In the first case (x_5, x_2), (x_2, x_3), $(x_3, x_5) \in E_m^T$, and in the second (x_5, x_4), (x_4, x_1), $(x_1, x_5) \in E_m^T$. Both possibilities contradict the transitivity of G_m^T, whence (x_1, x_2, x_3, x_4) are *not* in position 1. ▌

CLAIM A.2. *If there are vertices, x_5, x_6 for which $(x_1, x_2, x_3, x_4, x_5, x_6)$ are in an upper (lower) B-structure in G, it cannot happen that (x_1, x_2, x_3, x_4) are in an upper (lower) position 2.*

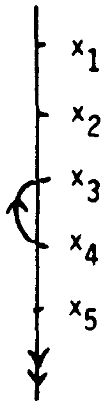

FIGURE 12

Proof. We will prove the claim only for the upper structure and position, since the proof for the other case is symmetric.

By the definition of the B-structure

$$(x_2, x_5), (x_5, x_3) \in E.$$

By that of position 2,

$$(x_3, x_2) \in E; (x_2, x_3) \in E_m^T.$$

This implies (x_2, x_5), $(x_5, x_3) \in E_m^T$. (Since only one of $\{(x_3, x_2), (x_2, x_5), (x_5, x_3)\} \subseteq E$ can be reversed in G_m^T and (x_3, x_2) is indeed reversed.) Consequently, $(x_5, x_4) \in E_m^T$ while $(x_4, x_5) \in E$ and $(x_2, x_1) \in E$, but $(x_1, x_2) \in E_m^T$. (See Fig. 13.) Therefore (x_1, x_2, x_5, x_4) are in position 1. By the definition of the B-structure, $(x_1, x_2, x_5, x_4, x_6)$ are in an A-structure, which contradicts Claim A.1. ▮

CLAIM A.3. *If there are vertices x_6, x_7, x_8 such that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ are in an upper (lower) C-structure, then $(x_1, x_2, x_3, x_4, x_5)$ cannot be in an upper (lower) position* 3.

Proof. Again we give the proof only for the upper structure and upper position since the lower structure and position are dealt with symmetrically. As in the preceding proofs, here we have (x_2, x_6), $(x_6, x_3) \in E_m^T$, whence (x_1, x_6), (x_6, x_4), $(x_4, x_5) \in E_m^T$, while (x_6, x_1), $(x_4, x_6) \in E$. That is, (x_1, x_6, x_4, x_5) are in an upper position 2. But, by the definition of the C-structure, $(x_1, x_6, x_4, x_5, x_7, x_8)$ are in an upper B-structure, in contradiction to Claim A.2. (See Fig. 14.) ▮

We now proceed to prove Claim A, that is, that $d \succ^m e \succ^m f$ is impossible. Assume the contrary, i.e., (a, b), (b, c), (a, c), (d, e), (e, f), $(d, f) \in E_m^T$. The two triangles abc and def, when "spanned" in the linear ordering $\succ^m$, may be in one of the following eight positions:

(1) *One of the triangles is "above" the other one*, i.e., $(c, d) \in E_m^T$ or $(f, a) \in E_m^T$. In this case, (a, c, d, f) are in position 1, but, by the definition of G, $(a, c, d, f, (1, 1, 1))$ are in an A-structure, a contradiction. (See Fig. 15.)

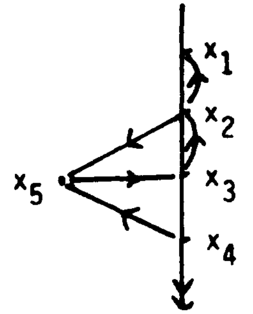

FIGURE 13

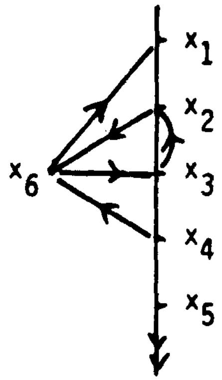

FIGURE 14

(2) *The triangle def is "covered" by abc*, that is, (a, d), $(f, c) \in E_m^T$. Note that (d, f), $(f, c) \in E_m^T$ while (f, d), $(c, f) \in E$, whence (a, d, f, c) are in a lower position 2. However, $(a, d, f, c, (1, 2, 1), (1, 2, 2))$ are in a lower B-structure, and hence this possibility has to be excluded. (See Fig. 16.)

(3) *The triangle abc is "covered" by def*, that is, (d, a), $(c, f) \in E_m^T$. In this case, (d, a), $(a, c) \in E_m^T$, but (a, d), $(c, a) \in E$, whence (d, a, c, f) are in an upper position 2. As $(d, a, c, f, (1, 3, 1), (1, 3, 2))$ are in an upper B-structure, *abc* cannot be "covered" by *def*. (See Fig. 17.)

(4) *The triangles "intersect" each other, where def is "higher"*, or (d, a), (a, f), $(f, c) \in E_m^T$. But (a, d), $(c, f) \in E$, so that (d, a, f, c) are in position 1, a contradiction to the fact that $(d, a, f, c, (1, 4, 1))$ are in an A-structure. (See Fig. 18.)

If none of the situations (1)–(4) occurs, the triangles are bound to "intersect" each other, with *abc* "higher" than *def*, that is, (a, d), (d, c), $(c, f) \in E_m^T$ (see Fig. 19). Describing the remaining possibilities, (5)–(8), we will not repeat this fact. We are therefore left with one of:

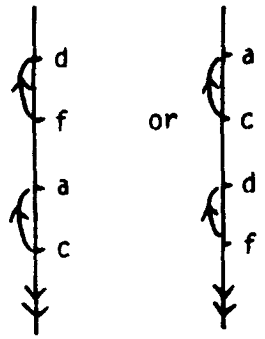

FIGURE 15

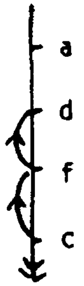

FIGURE 16

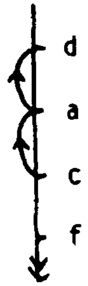

FIGURE 17

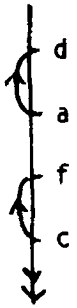

FIGURE 18

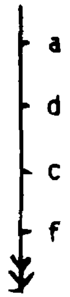

FIGURE 19

(5) *b is "above" d*, i.e., (a, b), $(b, d) \in E_m^T$. As $(c, d) \in E$, (a, b, d, c, f) are in a lower position 3. Since $(a, b, d, c, f, (1, 5, 1), (1, 5, 2), (1, 5, 3))$ are in a lower C-structure, this possibility contradicts Claim A.3. (See Fig. 20.)

(6) *b is "below" d, e is "below" c*, namely, (a, d), (d, c), (c, e), $(e, f) \in E_m^T$. Here (a, d, c, e, f) are in an upper position 3, while $(a, d, c, e, f, (1, 6, 1), (1, 6, 2), (1, 6, 3))$ are in an upper C-structure, again a contradiction. (See Fig. 21.)

(7) *b and e are "between" d and c, and e is "above" b*, or (a, d), (d, e), (e, b), $(b, c) \in E_m^T$. $(b, e) \in E$, hence (a, d, e, b, c) are in a lower position 3, while $(a, d, e, b, c, (1, 7, 1), (1, 7, 2), (1, 7, 3))$ are in a lower C-structure, which is impossible (See Fig. 22.)

(8) *b and e are "between" d and c, and b is "above" e*: (d, b), (b, e), $(e, c) \in E_m^T$. Recall that (b, d), $(c, e) \in E$, whence (d, b, e, c) are in position 1. However, since $(d, b, e, c, (1, 8, 1))$ are in an A-structure, this possibility must also be excluded. (See Fig. 23.)

It is easily seen that possibilities (1)–(8) exhaust all possible interrelations between the triangles *abc* and *def*, and as they were excluded one by one, we have proved Claim A, that is: it is impossible that $d \succ^m e \succ^m f$. ∎

We may now write

Claim B. *It is impossible that* $e \succ^m f \succ^m d$.

Claim C. *It is impossible that* $f \succ^m d \succ^m e$.

The proofs of these claims are identical to that of Claim A, where the vertices $\{(1, j, k)\}_{j,k}$ are replaced by $\{(2, j, k)\}_{j,k}$ and $\{(3, j, k)\}_{j,k}$, respectively. Since the remaining three permutations of *def* were proved impossible by the lemma, G is not $\frac{2}{3}$-decomposable. ∎

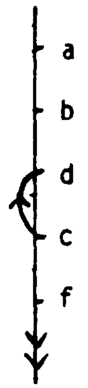

Figure 20

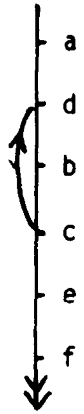

FIGURE 21

3.3. *Proof of the Insufficiency of the Diagonal Inequality*

In view of Subsection 3.2, the main point is:

LEMMA. *Let $G(N, E) \in \mathscr{E}$. Define*

$$Q_{ab} = \begin{cases} \frac{2}{3} & a \neq b,\ (a, b) \in E \\ \frac{1}{3} & a \neq b,\ (a, b) \notin E \\ 0 & a = b. \end{cases}$$

Then Q satisfies the diagonal inequality.

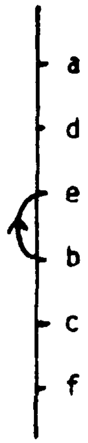

FIGURE 22

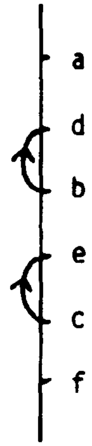

FIGURE 23

Proof. Let there be given two indices sets A, $B \subseteq N$:

$$A = \{a_i\}_{i=1}^{k}, \qquad B = \{b_j\}_{j=1}^{k}.$$

Denote

$$D = \sum_{i=1}^{k} Q_{a_i b_i}, \qquad S = \sum_{1 \leqslant i \neq j \leqslant k} Q_{a_i b_j}.$$

We wish to prove that, for all $1 \leqslant r \leqslant k-1$,

$$S \geqslant rD - \binom{r+1}{2}.$$

Distinguish between two cases:

Case (a). $k=2$, whence $r=1$. Here $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and

$$S = Q_{a_1 b_2} + Q_{a_2 b_1}, \qquad D = Q_{a_1 b_1} + Q_{a_2 b_2}.$$

If $S > 0$, then $S \geqslant \frac{1}{3}$. As $Q_{ab} \leqslant \frac{2}{3}$ for all a, $b \in N$, $D \leqslant \frac{4}{3}$. This implies

$$S - rD + \binom{r+1}{2} \geqslant \frac{1}{3} - \frac{4}{3} + 1 = 0.$$

If, on the other hand, $S = 0$, we necessarily have $Q_{a_1 b_2} = Q_{a_2 b_1} = 0$, whence $a_1 = b_2$, $a_2 = b_1$.

In this case, $D = Q_{a_1 b_1} + Q_{a_2 b_2} = 1$ and again $S - rD + \binom{r+1}{2} = 0 - 1 + 1 = 0$. That is, the diagonal inequality holds.

Case (b). $k > 2$. As $Q_{ab} \leqslant \frac{2}{3}$ for all a, $b \in N$, $D \leqslant \frac{2}{3}k$. To have a lower bound for S, we need:

Observation. Out of the $k(k-1)$ elements $Q_{a_i b_j}$ $(i \neq j)$, at most k may be zero.

Proof. Assume there are at least $(k+1)$ different pairs (i, j) for which $a_i = b_j$. Then there must be at least one index j for which there are $i_1 \neq i_2$ such that $a_{i_1} = b_j = a_{i_2}$. This contradicts the assumption that $|A| = k$. ∎

Since for $a \neq b$, $Q_{ab} \geqslant \frac{1}{3}$, we have

$$S \geqslant k(k-2) \cdot \tfrac{1}{3}$$

whence

$$\begin{aligned} S - rD + \tfrac{1}{2}r(r+1) &\geqslant \tfrac{1}{3}k(k-2) - \tfrac{2}{3}kr + \tfrac{1}{2}r(r+1) \\ &= \tfrac{1}{3}(k-r)(k-r-2) + \tfrac{1}{6}r(r-1). \end{aligned}$$

Again we distinguish between two cases:

Case (b.1). $r \leqslant k-2$, which implies $S - rD + \frac{1}{2}r(r+1) \geqslant 0$ immediately, and

Case (b.2). $r = k-1$, in which case

$$S - rD + \tfrac{1}{2}r(r+1) \geqslant -\tfrac{1}{3} + \tfrac{1}{6}(k-1)(k-2).$$

But $k > 2$ implies $(k-1)(k-2) \geqslant 2$, and hence

$$S - rD + \tfrac{1}{2}r(r+1) \geqslant 0,$$

so that Q satisfies the diagonal inequality. ∎

Our desired conclusion is:

CLAIM. *The diagonal inequality is not sufficient for a matrix Q to belong to* $\operatorname{conv}\{Y^{(m)}\}_m$.

Proof. Let G be the graph constructed in Subsection 3.2, which is not $\frac{2}{3}$-decomposable. Define Q as in the lemma, which also assures that Q satisfies the diagonal inequality. Note that were Q in $\operatorname{conv}\{Y^{(m)}\}_m$, G would have been $\frac{2}{3}$-decomposable. ∎

4. REMARKS

In this section we will show that all the conditions mentioned in Fishburn and Falmagne (1988) are insufficient.

4.1. *The Diagonal Inequality Generalizes the Cohen-Falmagne Conditions*

Proof. The Cohen–Falmagne conditions are the following: for every two subsets A, B with $|A|=|B|=m$ and $A\cap B=\varnothing$, and every 1–1 function f mapping A onto B,

$$\sum_{i\in A}\ \sum_{k\in B\setminus\{f(i)\}} Q_{ik}+\sum_{i\in A} Q_{f(i)\,i}\leqslant m(m-1)+1.$$

Given A, B, and f, let us enumerate the elements of A and B such that $f(a_j)=b_j$ for $i\leqslant j\leqslant m$. Then in our notation this condition can be rewritten as

$$\sum_{1\leqslant i\neq j\leqslant m} Q_{a_ib_j}+\sum_{i=1}^{m} Q_{b_ia_i}\leqslant m(m-1)+1$$

or

$$\sum_{1\leqslant i\neq j\leqslant m} (1-Q_{b_ja_i})+\sum_{i=1}^{m} Q_{b_ia_i}\leqslant m(m-1)+1$$

which is equivalent to

$$\sum_{1\leqslant i\neq j\leqslant m} Q_{b_ja_i}\geqslant\sum_{i=1}^{m} b_ia_i-1.$$

Note that this is exactly the diagonal inequality for $r=1$ ($k=m$ and A and B are, unfortunately, in reverse roles). ∎

4.2. *Fishburn's Conditions Are Insufficient*

Proof. Fishburn's conditions are of the form

$$\sum_{(i,j)\in C^+} Q_{ij}-\sum_{(i,j)\in C^-} Q_{ij}\leqslant 3k-2,$$

where $|C^+|=2\,|C^-|=4k-2$ and $k\geqslant 2$. Let us assume that $Q_{ij}\in[\frac{1}{3},\frac{2}{3}]$ for all i, j. Then it is easy to see that

$$\begin{aligned}\sum_{(i,j)\in C^+} Q_{ij}-\sum_{(i,j)\in C} Q_{ij}&\leqslant\tfrac{2}{3}(4k-2)-\tfrac{1}{3}(2k-1)\\&=2k-1\leqslant 3k-2.\end{aligned}$$

Hence these conditions are always satisfied for Q-matrices that do not contain numbers smaller than $\frac{1}{3}$ (equivalently, larger than $\frac{2}{3}$). In particular, the matrix Q of Section 3.3 above satisfies these inequalities, although it is not in $\operatorname{conv}\{Y^{(m)}\}_m$. ∎

Note that we have in fact shown that the diagonal inequality, the triangle inequality, Cohen–Falmagne conditions, and Fishburn's conditions taken together do not constitute a sufficient condition.

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RECEIVED: January 30, 1989