# Rationality and Zero Risk*

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**Abstract**

We adopt a definition of "rationality" as robustness to analysis: a mode of behavior is rational for a decision maker if she feels comfortable with it once it has been analyzed and explained to her. With this definition in mind, is it irrational to violate continuity axioms in one's stated preferences? Specifically, does it make sense to avoid *any* positive probability of a negative outcome, not matter how small? Or, if a decision maker states such a "zero risk" policy, does she mean what she says? We propose to study this question axiomatically, asking which modes of behavior correspond to such statements. The baseline model evaluates a lottery by its expected utility and an extra additive term that measures the cost of deviating from a "zero risk" choice. A generalized version allows for multiple sets of principles, where the cost of risking a set of principles is added to the expected utility of a lottery. Stronger assumptions imply that the cost of violating a set of principles is additive in the individual costs. We develop a comparative behavioral analysis that allows to make interpersonal comparisons about the relative importance of principles.

## 1 Introduction

According to the standard definition of "rationality" in economics, rational decision makers are logically omniscient, make decisions according to some classical model – such as von Neumann and Morgenstern (vNM, 1947) expected utility maximization – and typically ignore emotions in their (implicit) evaluation of outcomes. We find this definition unhelpful. First, if there were logically omniscient agents around, we would not have departments of mathematics or Chess tournaments, and probably would not be making online purchases. Second, there are situations in which decision makers who violate axioms of presumed-rationality insist on their choices despite being confronted with their analysis. Finally,

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many decision makers respond to affective payoffs and see no problem with that. In all these cases, we can dub these decision makers "irrational", but we cannot use our theories to change their behavior. A more practical definition of rationality, which we adopt here, would refer to such decision makers as rational as long as their behavior is robust to its own analysis.[1]

In this paper we focus on the rationality of some violations of continuity in decision making under risk. Expected utility theory (EUT) is continuous in probabilities: strict preferences between two lotteries are robust to sufficiently small changes in the probabilities defining one of them. Indeed, this is a natural mathematical condition that is satisfied by many formulas, even if they are not linear in probabilities. However, people and organizations can often *state* preferences that violate continuity (as we shall see below). Clearly, such preferences violate EUT. But are they rational according to the definition used here? Do people who state such preferences really mean what they say? Will they stick to their stated preferences once they are confronted with the analysis of their implications? Are the stated violations of continuity rational for them? Consider the following examples.

*Example 1.* A young couple is shopping for a car seat for their baby. They are on a tight budget, and realize that they may have to give up on some features that are a matter of convenience. But they would not compromise on safety. "We will not risk our baby's life for a few dollars", they tell the salesperson. The phrase "risking the baby's life" seems to correspond to "choosing a lottery with a positive probability of death". If we accept this reading, the verbal description of preferences is at odds with the continuity axiom of EUT. Specifically, it is a description of preferences that are discontinuous "at zero", that is, at $p = 0$ where $p$ is the probability of the horrendous outcome.

*Example 2.* The movie *Oppenheimer* has the following dialog:

> Gen. Leslie Groves: "Are you saying that there's a chance that when we push that button...we destroy the world?"
>
> J. Robert Oppenheimer: "The chances are near zero."
>
> Gen. Leslie Groves: "Near zero..."
>
> J. Robert Oppenheimer: "What do you want from theory alone?"
>
> Gen. Leslie Groves: "Zero would be nice.

The historical veracity of the dialog has been questioned.[2] Yet, it seems very plausible for policymakers to treat "some tiny probability" as qualitatively different from "zero probability".

[1]For further discussion of this definition and its variants, see Gilboa (1991, 2009, 2015) and Gilboa, Maccheroni, Marinacci, Schmeidler (2010).

[2]See "How Oppenheimer weighed the odds of an atomic bomb test ending Earth" (Washington Post, July 2023).

*Example 3.* In the context of the COVID-19 pandemic, a public health official claims, "Clearly, human lives come first. We will first minimize the risk of losing human lives and only then take into account economic considerations." Thus, the stated policy describes lexicographic preferences, which are discontinuous in probabilities. Indeed, as pointed out by Rubinstein (1998), when preferences are defined in a language, rather than represented by a mathematical function, lexicographic preferences might be rather natural.

An economist who hears the above might say, "I don't think you really mean what you say. You state preferences that are easily described in natural language, but if you were to truly think about what they mean, I bet you would choose otherwise." In other words, the economist argues that the stated, discontinuous preferences are irrational in the sense that the decision makers themselves would like to reconsider them, if they thought about them carefully enough.

But we could imagine, say, the parents in Example 1 countering, "Continuous or not, these are our preferences. We're willing to pay an extra $\$x$ to know that we have not put our child's life at *any* risk. Irrespective of the horrific outcome, which might well have a very low probability, the very fact that we chose to take the risk, while we could have chosen a zero-risk option, is costly. This cost is borne even if the dreadful outcome does not materialize." This justification is based on a moral principle, dictating that they not put their baby at risk. Alternatively, they might have discontinuity at zero because they believe that any probability $p > 0$ of their baby's death would suffice to deprive them of their peace of mind, along the lines of the Certainty Effect (Kahneman and Tversky, 1979). They might even be giving in to social pressure, which does not allow them to pick anything "but the safest". Whether the reason is moral, psychological, or social, there is a qualitative difference between choosing (a lottery with) $p = 0$ and choosing (a lottery with) $p > 0$, because these choices have *meaning*, and meaning may behave discontinuously at zero probability.

A similar argument can be made in Example 2. Indeed, politicians would generally find a qualitative difference between policies that do and that do not put the public at some risk – no matter how small the risk is. At the same time, our continuity-promoting economist might have a stronger case in Example 3. She might say, "OK, I understand that you attach a special meaning to zero probability, or to the act of choosing a non-zero probability, to be precise. But how can you seriously claim that you attach such meaning to each and every value in $[0, 1)$? The preferences you describe don't even have a numerical representation. I'm highly suspicious you're just following a slogan and not really thinking about what you say."

Note that in some examples discontinuity has to do with the choice (lottery) itself, in others – with the act of choosing it, and sometimes with both. In Example 1, for instance, one might argue that zero-risk isn't quite an option, and car accidents might always occur. Parents who pretend that this is not the case may be irrational: we could ask them, "Do you think that *any* car seat would drive the probability of the terrible outcome all the way down to zero?" and many would sadly acknowledge that the answer is in the negative. Thus, our analysis can force them to admit that their belief in zero-risk was an illusion. Yet, they can insist that it is still their duty to choose the lowest risk car seat. In Example 2, by contrast, a politician might say, "Yes, zero-risk is possible, if we don't develop nuclear weapons at all. Beyond that, I am concerned about the moral responsibility of choosing an option that doesn't minimize the probability of a catastrophe". We do not attempt to distinguish between discontinuities that are due to the nature of the lottery per se and those that are due to the act of choosing it. We consider stated preferences over presumably-observable choices, and it might be difficult to tell apart motivations that have to do with the alternatives and with the act of choice. Indeed, the decision maker herself may not always be able to draw this distinction.[3]

Which stated preferences are to be trusted and which are suspicious? In other words, which of the discontinuities described above are rational for the decision makers stating them? The question is obviously an empirical one, and it is unfortunately beyond the scope of this paper. Our goal here is only to contribute to the discussion by providing analytical tools to clarify what statements "really mean". We adopt the axiomatic approach, considering presumably-observable choices, and asking, which regularities these choices should satisfy in order to correspond to a certain description. Clearly, theoretical analysis cannot determine to what degree certain preferences are rational, and for whom. But it can help in translating abstract descriptions of preferences to concrete instances of choice, which may be more easily conceived of by decision makers.

We view the contribution of the axiomatic treatment as twofold. First, it can help us as theorists to judge the plausibility of discontinuous models, for positive and normative purposes alike. From a positive viewpoint, it aids in bridging stated preferences with observable behavior by characterizing the revealed preferences that correspond to individuals' linguistic statements. We can then use axioms as a predictive test, at least in principle, to check whether descriptions of preferences are credible or not. From a normative viewpoint, our axiomatic treatment offers a more encompassing notion of rationality, when acts of choice carry meaning, that otherwise would be missing from the standard EUT paradigm. More specifically, our characterizations should be thought of as a guide for choice behavior in the presence of moral principles. As expected, some of our axioms (those pertaining to weakenings of continuity) will not be testable in lab experiments as they involve infinitely many comparisons; yet, this limitation should not prevent economists or decision makers from using these axioms to consider hypothetical choices and to judge whether the discontinuity in preference stands to reason. The mind-experiments involved may be useful to tell apart sincere concerns from less-sincere slogans. For instance, Example 3 brings to

[3] Relatedly, some discontinuities would disappear if we allow the act of choice to be part of the outcome. Thus, a generalized outcome can be defined as a pair of a material outcome and the actor who is responsible for bringing it about. However, such outcomes are not directly observable.

the fore the case of stated lexicographic preferences. As argued before, this kind of statements are rather common – notably in political speeches – when preferences are expressed in natural language. One may therefore ask about the rationale behind these statements. The axiomatic method arms us with the necessary tools to address this question: the non-existence of a numerical representation of preferences in Example 3 suggests that actual behavior would not follow the stated lexicographic preferences. If "the risk of losing human lives" is perceived as a continuous variable, the statement made might be in conflict with actual decisions (if not vacuous). The second contribution of our axiomatic treatment addresses the following issue: if we allow for the possibility of discontinuous preferences, which model should we adopt? Of all the decision rules that violate continuity, which are more plausible to use for descriptive and for normative purposes? The axiomatic approach helps us in thinking about this problem, using hypothetical choice situations as a rhetorical device to argue for or against certain models.

### 1.1 A Standpoint on Rationality

We view this paper as part of a more general project, studying violations of classical axioms that can be regarded as rational. The term "rationality" is fraught with different meanings, and, clearly, whether a mode of behavior is rational or not depends on the definition one adopts. For example, Weber (1921) discussed four different notions of rationality and rationalization, whereas Simon (1976) distinguished between substantive and procedural rationality. As mentioned in the introduction, the definition we adopt here is rather pragmatic, and it is based on the question: can theoretical analysis convince decision makers that they would have liked to behave differently?

Whereas for many economists "rationality" is defined by the classical vNM axioms, psychologists and economists have long noted that decision makers' behavior is often non-linear in probabilities, especially near zero (Preston and Baratta, 1948, Allais, 1953, Edwards, 1954, Kahneman and Tversky, 1979). Moreover, some of the alternative theories that have been suggested to accommodate such behavior involve discontinuities (Gilboa, 1988, Jaffray, 1988)[4]. Are these behaviors rational according to our definition? We suggest that an axiomatic treatment is needed to conduct this rationality test.

In Minardi ⓡ Gilboa ⓡ Wang (2023) we employ this approach to study continuity in consumer choice. Whereas classical theory assumes that preferences are continuous in quantities of goods, there are cases in which households may systematically violate this assumption. For example, a vegetarian consumer would exhibit discontinuities of preferences at zero quantity of meat: the tiniest amount of meat renders a bundle non-vegetarian and

[4]In these models there is also a discontinuity at zero probability, but this is at the worst outcome within each lottery. Specifically, a lottery is evaluated by some function of its expected utility as well as the worst outcome in its support. By contrast, in the present paper we study discontinuity at zero probability of some pre-defined outcomes. That is, the discontinuity emerges from something that is inherent to the outcome (such as losing one's baby) rather than from its relative ranking in the support of the lottery.

can change behavior in a non-negligible way. Is such a discontinuity rational? We might put it to a test, and imagine a dialog with the vegetarian consumer, in which we (the analysts) try to argue that a single molecule of meat can't possibly make a bundle less desirable in noticeable way. Perhaps some consumers will be convinced that their zero-meat principle is indeed foolish. But others are likely to respond, "Yes, of course, once we have even a little bit of meat, I feel that I lose something qualitatively. Avoiding meat consumption is a matter of principle for me. Consuming a non-vegetarian dish changes the meaning of the act of consumption, and the assignment of meaning to acts behaves in a discontinuous way." Thus, such a consumer might insist on her stated, discontinuous preferences, and whether we dub her rational or not, we have to admit that this type of behavior is here to stay. The axiomatic analysis of such preferences is supposed to help decision makers ask themselves, "Do I feel comfortable with this type of behavior?"

Along similar lines, the present paper attempts to aid the analysis of discontinuous behavior when making choices under risk. As in the case of discontinuities in quantities of goods, discontinuities in probabilities of outcomes cannot be directly observed in finite databases of choices. But they can be described in a natural, legal, or formal language. Is it rational to adopt such preferences? Do decision makers who state preferences in a way that implies discontinuities in probabilities like their own preference, and do they really mean what they say? We offer the axiomatic analysis as a way to address this question: characterizing decision patterns that can be described in discontinuous ways may aid decision makers in figuring out whether they wish to adopt such preferences despite the violation of vNM's continuity axiom.

**Outline.** We ask, which of the von Neumann-Morgenstern (vNM) axioms need to be relaxed, and how, to allow for certain types of discontinuity. We focus on zero-risk, that is, on preferences that are discontinuous at zero probability on a subset of outcomes, intuitively thought of as the negative ones. For concreteness, we may think of the negative outcomes as relating to a given principle, as in the baby seat example (Example 1). The next section studies choice behavior when the agent wishes to abide by a certain principle, and faces a cost when she takes the risk of violating it. In this single-value setting, we present different weakenings of the Independence axiom and provide axiomatic characterizations that clarify the content of such weakenings, as well as their relationship. Section 3 extends this baseline model to the more complex case of choice behavior when multiple values are at stake. Section 3.4 discusses the significance and applied relevance of our results as modelling tools. All proofs are contained in the Appendix.

## 2 A Single Principle

For this section we assume as given a principle (or criterion, value) by which outcomes are classified. While this single-principle setting allows for easier illustration, the key insights we gain here turn out to be applicable also for the general, multiple-principle case, as shown in Section 3.

### 2.1 Setting

Let $X$ be a *finite* set of outcomes and $X_0 \subset X$ be a proper, nonempty subset of $X$. Let $L = \Delta(X)$ be the set of vNM lotteries on $X$. We consider a binary relation $\succsim$ on $L$, referred to as the preference, with the symbols $\succ$ and $\sim$ standing for its asymmetric and symmetric component, respectively. The preference is assumed to be complete and transitive.

> **A1. Weak Order:** $\succsim$ is complete and transitive on $L$.

To streamline notation, the preference rank $E \succsim Q$ (resp. $E \succ Q$) between a set $E \subset L$ of lotteries and a lottery $Q \in L$ indicates that $P \succsim Q$ (resp. $P \succ Q$) for all $P \in E$. Likewise, the notations $Q \succsim E$ and $Q \succ E$ carry analogous meaning. For any two sets of lotteries $E, F \subset L$, the preference rank $E \succsim F$ means $P \succsim F$ for all $P \in E$, or, equivalently, $P \succsim Q$ for all $P \in E$ and $Q \in F$.

Let $L_0 \subset L$ denote the subset of lotteries whose support is in $X_0$ and define $L_1 = L \backslash L_0$. We think of the lotteries in $L_0$ as "safe", or "zero risk", and the lotteries in $L_1$ as "unsafe". Our focus is on how preferences treat lotteries in $L_0$ and $L_1$ in a different way, and on delineating the formal meaning for this difference. To start with, we assume that the standard vNM axioms are respected on the domains $L_0$ and $L_1$, separately. Notice that, for all $P, Q \in L$ and real number $\alpha \in [0, 1]$, the mixture notation $\alpha P + (1 - \alpha)Q$ stands for the lottery that assigns probability $\alpha P(x) + (1 - \alpha)Q(x)$ to any outcome $x \in X$.

The following axioms are standard Archimedean and Independence properties imposed on $L_0$ and $L_1$, separately.

> **A2. Restricted Continuity:** For every $P, Q, R \in L$, if $P \succ Q \succ R$, and if $P, Q, R \in L_0$ or $P, Q, R \in L_1$, then there exist $\alpha, \beta \in (0, 1)$ such that
>
> $$\alpha P + (1 - \alpha)R \succ Q \succ \beta P + (1 - \beta)R.$$

> **A3. Restricted Independence**: For every $P, Q, R \in L$, if $P, Q, R \in L_0$ or $P, Q, R \in L_1$, then for every $\alpha \in (0, 1)$,
>
> $$P \succ Q \Longrightarrow \alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)R.$$

Since the mixture notion qualifies $L_0$ and $L_1$ as mixture sets in the sense of Herstein and Milnor (1953), A1-A3 hold if and only if $\succsim$ admits an expected utility representation on $L_0$ and $L_1$, separately.[5] More precisely, for any generic function $u$ on $X$ or on a subset thereof, we denote its expectation by:

$$U(P) = Eu(P) = \sum_{x \in X} P(x)u(x). \tag{2.1}$$

Under A1-A3, there exist functions $u_0$ on $X_0$ and $u_1$ on $X$ such that, for every $P, Q \in L_i$ with $i \in \{0, 1\}$,

$$P \succ Q \Longleftrightarrow U_i(P) > U_i(Q).$$

Both $u_0$ and $u_1$ are unique up to affine transformation.

For easy reference to the functions $u_0$ and $u_1$, we maintain A1-A3 throughout this section, though A2 and A3 are subsumed by the stronger axioms we are about to introduce.

## 2.2 Preference between $L_0$ and $L_1$

Throughout this section, we will make use of the following notion of preference overlapping.

**Definition 1.** *We define the* preference overlapping *between the domains $L_0$ and $L_1$ as the set*

$$O_\sim = \{(P, Q) : P \sim Q, P \in L_0, Q \in L_1\}.$$

*Similarly, we define $O_{0,\sim} = \{P \in L_0 : \exists (P, Q) \in O_\sim\}$ and $O_{1,\sim} = \{Q \in L_1 : \exists (P, Q) \in O_\sim\}$ as the overlapping parts of $L_0$ and $L_1$, respectively.*

The axioms A2 and A3 are silent about the structure of $O_\sim$, because they are only applicable to lotteries belonging to the same domain. But, as both $L_0$ and $L_1$ are convex and the utilities $U_0$ and $U_1$ are affine, it is reasonable to expect $O_{0,\sim}$ and $O_{1,\sim}$ to be convex, as well. This can be achieved by a slight strengthening of the Restricted Continuity axiom.

> **A2*. Semi-Restricted Continuity:** For every $P, Q, R \in L$, if $P \succ Q \succ R$, and if $P, R \in L_0$ or $P, R \in L_1$, then there exist $\alpha, \beta \in (0, 1)$ such that
>
> $$\alpha P + (1 - \alpha)R \succ Q \succ \beta P + (1 - \beta)R.$$

While the bounding lotteries $P$ and $R$ are still required to be in the same domain, just like A2, the bounded lottery $Q$ can now belong to a different domain. The axiom rules out, for instance, the case $P \in L_0$ and $R \in L_1$, for which any mixture between $P$ and $R$ risks the principle and may be deemed much worse than both $P$ and $Q$. However, as long as $P$ and $R$ are in the same domain, their mixtures do not introduce a new layer of concern over the principle. Hence, the usual normative argument for continuity is maintained - the mixture presents a minimal variation on $P$ when $\alpha$ is close to 1, or on $R$ as $\beta$ approaches 0.

[5] This is formally stated in Lemma 1, Section 5.1.

Under A2*, the preferences on $L_0$ and $L_1$ overlap in a continuous way. This restricts the class of monotone transformations of $U_0$ that represent preferences on $L_0$ to the case of continuous monotone transformations, as we formalize next.

**Proposition 1.** *Assume A1-A3. Then, A2* holds if and only if there exist functions $u_0$ on $X_0$ and $u_1$ on $X$, and a continuous strictly increasing function $g : U_0(L_0) \to \mathbb{R}$ such that, for every $P \in L$,*

$$P \mapsto \begin{cases} g(U_0(P)), & P \in L_0 \\ U_1(P), & P \in L_1 \end{cases}$$

*represents $\succsim$. The functions $u_0$ and $u_1$ are unique up to affine transformation, and, given $u_0$ and $u_1$, function $g$ is unique on $U_0(O_{0,\sim})$.*

Turning to A3, the Restricted Independence axiom, notice that it cannot be used when lotteries from different domains are involved, and, like A2, this reflects the fact that the mixture operation can have asymmetric effects on lotteries' safety. For example, if $P \succ Q$ when $P \in L_0$ and $Q \in L_1$, it is possible that the preference would reverse when we mix the two with $R \in L_1$: consider the mixtures $\frac{1}{2}P + \frac{1}{2}R$ and $\frac{1}{2}Q + \frac{1}{2}R$. Both are in $L_1$, that is, they are unsafe. However, as $Q$ was unsafe to begin with, its status doesn't change by mixing. By contrast, $P$ used to be a safe lottery, and the mixing with $R$ made it unsafe. As a result, there is no reason that the preference $P \succ Q$ would carry to the mixtures (of each of $P, Q$ with $R$).

Yet, we might limit such asymmetries and obtain a sensible condition even in the case of mixtures of safe and unsafe lotteries. To start with, notice that if both $P$ and $Q$ are safe lotteries, mixing them with an unsafe lottery would make both of them unsafe. If the damage of becoming unsafe is the same for $P$ and $Q$, then we can expect the preference between them to be preserved after the mixture. We are hence led to the following stronger version of A3.

> **A3*. Semi-Restricted Independence**: For every $P, Q, R \in L$, if $P, Q \in L_0$ or $P, Q \in L_1$, then for every $\alpha \in (0, 1)$,
>
> $$P \succ Q \Longleftrightarrow \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R.$$

As A3* still does not address the case where $P \in L_0$ and $Q \in L_1$, it gives no extra structure to the preference overlapping $O_\sim$. However, we can further build on the idea of independence to shed light on this issue as well. Concretely, under the standard Independence axiom, for any four lotteries $P, Q, R, S$, if $P \sim Q$ and $R \sim S$, then $\frac{1}{2}P + \frac{1}{2}R \sim \frac{1}{2}Q + \frac{1}{2}S$. The immediate proof uses Independence twice, showing first that $\frac{1}{2}P + \frac{1}{2}R \sim \frac{1}{2}Q + \frac{1}{2}R$ and then that $\frac{1}{2}Q + \frac{1}{2}R \sim \frac{1}{2}Q + \frac{1}{2}S$. In our case, the same argument would apply if all lotteries involved are in the same space ($L_0$ or $L_1$). But if the indifference $P \sim Q$ (or $R \sim S$) is across spaces, A3 does not allow us to proceed with the argument.

Indeed, if $P, Q$ (with $P \sim Q$) belong to different spaces while $R, S$ (with $R \sim S$) belong to the same space, neither $\frac{1}{2}P + \frac{1}{2}R \sim \frac{1}{2}Q + \frac{1}{2}S$ nor $\frac{1}{2}P + \frac{1}{2}S \sim \frac{1}{2}Q + \frac{1}{2}R$ seems very convincing. In particular, if $P \in L_0$ while $Q, R, S \in L_1$, mixing $P$ with either $R$ or $S$ involves venturing out of (the safe) $L_0$ into (the unsafe) $L_1$, while the other mixture does not switch spaces. In such a case the original equivalences, $P \sim Q$ and $R \sim S$, need not imply the equivalence of the mixtures. Similar asymmetry would arise if, say, $P, Q, R \in L_0$ but $S \in L_1$.

However, there is some logic in expecting the equivalence of the mixtures to hold if the mixing deals with the spaces in a "balanced" way. This suggests that an additional axiom might be reasonably imposed. Note, however, that there are now two possible versions of the axiom: one in which each mixing is done within a single space, and another in which both mixing operations cross spaces. We are therefore led to state two separate properties. We first state a weakening of independence where the mixing operator is applied within each space:

> **A4. Intra-Space Independence**: For $P, Q \in L_0$ and $P', Q' \in L_1$, if $P \sim P'$ and $Q \sim Q'$, then $\frac{1}{2}P + \frac{1}{2}Q \sim \frac{1}{2}P' + \frac{1}{2}Q'$.

If an agent exhibits indifference between some safe lotteries and some unsafe ones, then such an indifference will be preserved when comparing the mixture of the safe lotteries with the mixture of the unsafe ones. Behaviorally, it states that if the hedonic values of the unsafe lotteries counterbalance the risk of violating the principle so that the agent is indifferent in the original comparisons between safe and unsafe lotteries, then such counterbalance will be preserved when comparing the mixtures within each space. The next result clarifies the gains of imposing Intra-Space Independence in terms of utility representation. Note that even though the utility function $u_1$ is derived from the preference on $L_1$, it is actually defined over the whole set of outcomes $X$, and hence $U_1$ can be used to evaluate lotteries in $L_0$ according to equation (2.1).

**Proposition 2.** *Assume A1-A3. Then, the following statements hold:*

*(1) A3* holds if and only if $u_0$ and $u_1$ can be chosen to be the same on $X_0$.*

*(2) Assuming A2*, A4 holds if and only if the function $g(\cdot)$ in Proposition 1 can be chosen to be affine.*[6]

Proposition 2 shows that Axiom A3* only concerns the relationship between $u_0$ and $u_1$: it guarantees that $u_0$ and $u_1$ can be chosen to be the same, and, hence, that the hedonic values of both safe and unsafe lotteries can be computed using the same calibration of utility. On the other hand, Axiom A4 only restricts the functional form of the bridging function $g(\cdot)$ of Proposition 1 to be affine.

[6] A4 holds if and only if $g(\cdot)$ is affine on $U_0(O_{0,\sim})$. However, $g(\cdot)$ still has flexibility on $U_0(L_0 \backslash O_{0,\sim})$.

The second version of the independence axiom applies the mixing operator across the spaces $L_0$ and $L_1$, as shown next:

**A5. Inter-Space Independence**: For $P, Q \in L_0$ and $P', Q' \in L_1$, if $P \sim P'$ and $Q \sim Q'$, then $\frac{1}{2}P + \frac{1}{2}Q' \sim \frac{1}{2}P' + \frac{1}{2}Q$.

Axiom A5 can be viewed as the symmetric version of A4: while the antecedent is the same, the consequent maintains that indifference is preserved between the mixtures of safe and unsafe lotteries. Thus, A4 compares safe mixtures with risky ones, whereas in A5 the two mixtures become both risky. Behaviorally, A5 prescribes that in compounding safe and risky lotteries in the mixtures, the hedonic values generated by the unsafe components counteract the risk of violating the principle in the same way in which this counterbalance takes place in the individual indifferences.

Note that A4 and A5 become vacuous when the preference overlapping $O_\sim$ is empty. This can happen in two cases. If the agent considers any safe lottery strictly more valuable than any risky lottery: such an agent always adheres to the principle of not taking any risk. Or, if the agent considers any risky lottery strictly better than any safe one: in this case, we can think of the outcomes in $X \backslash X_0$ as positive ones, where the discontinuity occurs near certainty of the negative outcomes, $X_0$.[7] Furthermore, for the full potential of these axioms—especially of A5—we shall often require the preference overlapping $O_\sim$ to be *nontrivial*, that is:

**Definition 2.** *We say that $O_\sim$ is* nontrivial *if there exist $(P, P')$ and $(Q, Q')$ in $O_\sim$ such that $P \succ Q$.*

Definition 2 guarantees the existence of pairs $(P, P')$ and $(Q, Q')$ in $O_\sim$ across which the decision maker holds a strict preference, that is, $P' \sim P \succ Q \sim Q'$. This ensures that the overlap between $L_0$ and $L_1$ admits strict preferences and, hence, does not consist only of trivial indifferences. Armed with this notion of nontriviality, we can state a preliminary result that clarifies the relation between these three distinct weakenings of independence and, in particular, highlights the strength of A5 over A3* and A4.

**Proposition 3.** *Assume that A1, A2*, and A3 hold, and $O_\sim$ is nontrivial. Then A5 implies A3* and A4.*

We are now ready to characterize the content of Inter-Space Independence (A5) in terms of utility representation. We view it as our first main result providing a structured and tractable model of decision makers that have a preference for "zero risk".

**Theorem 1.** *The following statements are equivalent:*

[7] For example, a military operation would typically involve some risk to human lives, but sending troops to a certain death may feel different than putting their lives at risk.

*(i) The preference $\succsim$ satisfies A1, A2*, and A3. Moreover, A5 holds if $O_\sim$ is nontrivial and otherwise A3* holds.*

*(ii) The preference $\succsim$ can be represented as follows: for every $P \in L$,*

$$P \mapsto \begin{cases} U(P) + b, & P \in L_0 \\ U(P), & P \in L_1 \end{cases}, \tag{2.2}$$

*where $U(\cdot)$ denotes the expected utility for some utility function $u : X \to \mathbb{R}$ and $b \in \mathbb{R}$.*

*Moreover, $u$ is unique up to affine transformation and, when $O_\sim$ is nonempty, $b$ is unique given $u$.*

Theorem 1 shows that Axiom A5 delivers more than A3* and A4 combined together, since the function $g(\cdot)$ now has to be a translation. Representation (2.2) evaluates lotteries by means of two parameters, the utility function, $u$, and the cost, $b$, that the agent attaches to the risk of obtaining an (undesirable) outcome in $X \backslash X_0$. Intuitively, $L_0$ and $L_1$ are like two paper slips glued together, with the glued part standing for the overlapping $O_\sim$. Thus, a risky lottery is evaluated according to a standard expected utility criterion with Bernoulli utility given by some $u$. For a safe lottery, instead, the overall utility is given by the expected utility component (with same $u$), that reflects purely hedonic aspects, plus a premium, quantified by $b$, that reflects the value from satisfying the principle of not taking any risk. Naturally, the parameter $b$ can be thought of as a premium for choosing a safe lottery or, equivalently, as a cost incurred for choosing a risky lottery. Under either interpretations, the parameter $b$ captures the agent's preference for "zero risk", and the additive form allows us to identify it uniquely and to distinguish it from the hedonic component. The simplicity of the utility structure, together with its uniqueness properties, makes the model highly tractable, as shown by our comparative analysis later on.

**An alternative characterization.** We now approach the model in (2.2) from a different behavioral angle. This alternative characterization will be useful to extend our results to the case of multiple principles in the next section. To start, if two lotteries are indifferent to the agent and both of them assign the same probability to a safe outcome $x$, we should be able to expect this indifference to be intact after $x$ is changed to another safe outcome $x'$. The reason is two-fold. First, changing a safe outcome to another safe outcome does not alter the nature of the lottery, that is, it stays in $L_0$ or $L_1$. Second, we expect the hedonic utility of a safe outcome to be the same across any two lotteries. We are hence led to the following axiom.

**A6.** **$L_0$-Cancellation**: For every $P \in L_0$, $Q \in L_1$, $R, R' \in L_0$, and $\alpha \in (0, 1)$,

$$\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)R \implies \alpha P + (1-\alpha)R' \sim \alpha Q + (1-\alpha)R'.$$

That is, whether $P$ is indifferent to $Q$ should not depend on a common zero-risk component $R$ shared by the two lotteries. This axiom also carries the idea of independence, and, indeed, it is implied by the standard independence axiom. It is very similar to the weakening of Independence used in Maccheroni, Marinacci, and Rustichini (2006), though, as shown in our proofs, the mathematical underpinnings are quite different.

If $L_0$ and $L_1$'s preference overlapping $O_\sim$ is empty, A6 is immaterial and in particular $U_0$ and $U_1$ may disagree arbitrarily on $L_0$. If $O_\sim$ is nonempty but $U_0(O_{0,\sim})$ is a singleton (that is, for all $(P,P'),(Q,Q') \in O_\sim$, we have $P \sim Q$), A6 is again immaterial if, for all $(P,P') \in O_\sim$, the supports of $P$ and $P'$ are disjoint. But if the supports intersect for some $(P,P') \in O_\sim$, then we can take any outcome $x$ in this intersection and $P \sim P'$ can be re-written as

$$\alpha\hat{P} + (1-\alpha)x \sim \alpha\hat{P}' + (1-\alpha)x$$

for some $\hat{P} \in L_0$ and $\hat{P}' \in L_1$, and $\alpha \in (0, min\{P(x), P'(x)\}]$. By A6 we have $\alpha\hat{P} + (1-\alpha)Q \in O_{0,\sim}$ for all $Q \in L_0$, and, by the linearity of $U_0$ and $U_0(O_{0,\sim})$ being a singleton, we must have $x \sim Q$ for all $Q \in L_0$, indicating total indifference on $L_0$.

A6 delivers much more when $O_\sim$ is nontrivial. Roughly speaking, A6 indicates that lotteries in $L_0$ are locally evaluated in the same way by $U_0$ around $P$ and $U_1$ around $P'$. Due to the linearity of expected utility, A6 implies that $U_0$ and $U_1$ have the same evaluation over the entire $L_0$, which is also what A3* demands. And, by integrating this local property across $O_\sim$, A6 can also yield the global properties of A4 and A5. We hence have the following result.

**Proposition 4.** *Assume that A1, A2*, and A3 hold and $O_\sim$ is nontrivial. Then, A6 implies A3* and A4, and it is equivalent to A5.*

Proposition 4 is similar to Proposition 3: the only key novelty comes from the equivalence between Inter-Space Independence (A5) and $L_0$-Cancellation (A6). This means that the axiomatic underpinning of Representation (2.2) rests on either of the two properties. Both axioms can be easily tested in a lab experiment, and have a clear behavioral interpretation. A6 is an immediate weakening of Independence, and it is, perhaps, cognitively less demanding, from both a descriptive and normative perspectives.

### 2.3 Positive Cost and Comparative Analysis

Representation (2.2) does not impose any restriction on the sign of the parameter $b$. Clearly, assuming standard continuity throughout $L$ would imply $b = 0$. However, under Semi-Restricted Continuity A2*, we can state conditions under which $b$ in representation (2.2) has to be positive or negative. For example, in the extreme case where all safe lotteries are deemed strictly better than any unsafe lottery (i.e., $L_0 \succ L_1$), it is easy to see that $b \geq 0$. Similarly, in the opposite extreme case where $L_1 \succ L_0$, it must be that $b \leq 0$. Consider the most interesting case where the preference overlapping $O_\sim$ is nonempty so that $b$ is uniquely identified, thereby allowing us to draw sharper inferences. In particular, let $P \sim Q$ for some $P \in L_0$ and $Q \in L_1$. Then, mixing $P$ and $Q$ can only be less valuable than $P$ alone since the mixture alters the zero-risk nature of $P$ without bringing any hedonic benefits. We formalize this intuition below and show that it behaviorally characterizes the case where $b > 0$.

> **Positivity**: For every $P \in L_0$ and $Q \in L_1$, $P \sim Q$ implies $P \succ \alpha P + (1-\alpha)Q$ for every $\alpha \in (0, 1)$.

**Proposition 5.** *Suppose that $\succsim$ admits representation (2.2) and $O_\sim$ is nonempty. Positivity holds if and only if $b > 0$.*

For agents whose preferences are represented by (2.2) with $b > 0$, a natural question that arises is how agents may differ in their attitudes toward "unsafe" lotteries. The following definition formalizes, in behavioral terms, what it means for one agent to be more averse to selecting "unsafe" lotteries than another agent.

**Definition 3.** *Let $\succsim_1$ and $\succsim_2$ be two preferences on $L$. Then, $\succsim_1$ is more inclined to zero-risk than $\succsim_2$ if the following conditions are satisfied:*
*(i) for every $Q, S \in L_1$, $Q \succsim_1 S$ if and only if $Q \succsim_2 S$,*
*(ii) for every $P \in L_0$ and $Q \in L_1$, $P \succsim_2 Q$ implies $P \succsim_1 Q$.*

Condition (i) ensures that agents have the same preferences over $L_1$. This is a prerequisite for making meaningful comparisons between agents. Condition (ii) is the key comparative notion: suppose that Agent 2 prefers some safe lottery, $P$, to some other risky one, $Q$. If Agent 1 is more inclined to zero-risk than Agent 2, then she will, a fortiori, prefer $P$ to $Q$.

The following proposition characterizes the comparative notion of *inclination to zero-risk* in terms of the representation provided in Proposition 2. The notation $u_1 \approx u_2$ stands for $u_1 = \lambda u_2 + d$ for $\lambda > 0$ and $d \in \mathbb{R}$.

**Proposition 6.** *Let $\succsim_1$ and $\succsim_2$ be two preferences on $L$ that admit representation (2.2) with $(u_1, b_1)$ and $(u_2, b_2)$, respectively. Suppose further that $\succsim_2$ has a nonempty preference overlapping. Then, the following conditions are equivalent:*
*(i) $\succsim_1$ is more inclined to zero-risk than $\succsim_2$,*
*(ii) $u_1 \approx u_2$ and, normalizing the representations to have $u_1 = u_2$, we also have $b_1 \geq b_2$.*

Proposition 6 states that more-inclined-to-zero-risk preferences are characterized by greater parameters $b$, up to a normalization. For concreteness, focus on the case $b_1, b_2 > 0$, in which both individuals prefer zero-risk. Thus, $b_i$ is the cost incurred by individual $i$ should she choose a positive-risk lottery, where $b_i$ is measured on the expected utility scale. Comparing two such individuals, $\succsim_1$ is more inclined to zero-risk than $\succsim_2$ is, after having normalized their utility functions, we have a higher cost for individual 1 than for 2. In short, the parameter $b$ can thus be thought of as an index of inclination to zero-risk.

## 2.4 Infinitely Many Outcomes

We now extend the analysis to the case where the set $X$ may contain infinitely many outcomes (e.g., all monetary outcomes), and, correspondingly, the preference $\succsim$ is defined over the set $L = \Delta_0(X)$ of simple lotteries (that is, lotteries with finitely many outcomes in the support). When the number of outcomes is infinite, utility $U_1$ in representation (2.2) can be unbounded and, given a finite $b$, every lottery in $L_0$ must be less preferred than some lottery in $L_1$. For this reason, we need a preference condition to convey the idea that "every principle has a price", and, as we shall see, this is the only extra care we need to take in extending the model to the case of infinitely many outcomes.

We start by commenting that all the axioms stated for the finite-outcome case are still applicable here. In what follows, we try to formulate unbounded utility in terms of preference.

**Definition 4.** *A sequence of lotteries $\{P_n\}_{n=1}^{\infty}$ entirely contained in either $L_0$ or $L_1$ is* ascending *if $P_2 \succ P_1$ and $\frac{1}{2}P_{n+1} + \frac{1}{2}P_{n-1} \succsim P_n$ for all $n \geq 2$, and it is* descending *if $P_1 \succ P_2$ and $P_n \succsim \frac{1}{2}P_{n+1} + \frac{1}{2}P_{n-1}$ for all $n \geq 2$.*

When the preference $\succsim$ is linear on $L_0$ and $L_1$, as guaranteed by the Restricted Independence axiom A3, the relation $\frac{1}{2}P_{n+1} + \frac{1}{2}P_{n-1} \succsim \frac{1}{2}P_n + \frac{1}{2}P_n$ suggests that the difference between $P_{n+1}$ and $P_n$ more than compensates that between $P_n$ and $P_{n-1}$. In the same spirit, given $P_2 \succ P_1$, the "preference incremental" from $P_n$ to $P_{n+1}$ should be higher than that from $P_1$ to $P_2$, for all $n \geq 2$. Hence, an ascending sequence of lotteries is an indication of a utility function that is not bounded from above. Similarly, a descending sequence of lotteries signals a utility function that is not bounded from below. We are therefore led to the following condition.

> **Archimedeanity:** For any ascending sequence of lotteries $\{P_n\}_{n=1}^{\infty}$, there cannot exist a $Q$ such that $Q \succsim P_n$ for all $n$. And, for any descending sequence of lotteries $\{P_n\}_{n=1}^{\infty}$, there cannot exist a $Q$ such that $P_n \succsim Q$ for all $n$.

If we have an ascending sequence $\{P_n\}_{n=1}^{\infty}$ in $L_1$, Archimedeanity implies that for any zero-risk lottery $Q$, there always exist some $P_n$ in the sequence such that $P_n \succsim Q$. Conceptually, it rules out the scenario where values consideration overwhelms hedonic appeal no matter how large the latter is.

**Proposition 7.** *The following two statements are equivalent:*

*(a) The preference $\succsim$ satisfies A1, A2*, and A3. If $O_\sim$ is nontrivial, then A5 holds, otherwise A3* holds. Furthermore, if $O_\sim$ is empty, Archimedeanity holds.*

*(b) The preference $\succsim$ can be represented as follows: for every $P \in L$,*

$$P \mapsto \begin{cases} U(P) + b, & P \in L_0 \\ U(P), & P \in L_1 \end{cases}, \tag{2.3}$$

*where $U(\cdot)$ denotes the expected utility for some utility function $u : X \to \mathbb{R}$ and $b \in \mathbb{R}$.*

*Moreover, $u$ is unique up to affine transformation and, when $O_\sim$ is nonempty, $b$ is unique given $u$.*

Note that Archimedeanity is invoked only when $O_\sim$ is empty: in this case, we need to rule out the case where the agent prefers any risky lottery to any safe lottery (or vice versa). When $O_\sim$ is non-empty, Archimedeanity is superfluous and the statement of Proposition 7 is exactly the same as that of the finite case (see Theorem 1). Hence, the key behavioral conditions (mostly A5) remain unchanged.

# 3 Multiple Principles

It is not uncommon for economic agents to have more than one principle. In verbal discussions people tend to espouse many principles, each of which sounds convincing on its own. The question then arises, what will they do when these principles are in conflict with each other and/or with hedonic well-being? This section extends the previous single-principle analysis to this more general setup.

## 3.1 Setting

Let $X$ denote a finite collection of outcomes and $L = \Delta(X)$ be the set of vNM lotteries on $X$. Furthermore, let $\mathcal{K} = \{1, 2, ..., K\}$ be a family of $K$ principles where each principle $k \in \mathcal{K}$ is associated with a set of outcomes $X_k \subset X$ that violate principle $k$. The set $X_0 = X \backslash (\cup_{k \in \mathcal{K}} X_k)$ comprises the "good outcomes" that do not violate any principle. For any subset $\mathcal{I} \subset \mathcal{K}$ of principles, let

$$L_{\mathcal{I}} = \{P \in \Delta(X) : P(X_k) > 0, \forall k \in \mathcal{I}; P(X_{k'}) = 0, \forall k' \in \mathcal{I}^c\}$$

denote the set of lotteries that risk violating *all and only* those principles in $\mathcal{I}$. When $\mathcal{I} = \varnothing$, $L_\varnothing = \Delta(X_0)$ is the set of zero-risk lotteries. Notice that lotteries in $L_{\mathcal{I}}$ are also allowed to assign positive probabilities to the good outcomes $X_0$. In general, every $P \in L_{\mathcal{I}}$ has its support contained in $X_0 \cup (\cup_{k \in \mathcal{I}} X_k \backslash \cup_{k' \in \mathcal{I}^c} X_{k'})$. Moreover, $L_{\mathcal{I}} \cap L_{\mathcal{I}'} = \varnothing$ for all $\mathcal{I} \neq \mathcal{I}'$, and $\Delta(X) = L_\varnothing \cup (\cup_{\mathcal{I} \subset \mathcal{K}} L_{\mathcal{I}})$.

We shall impose the following structural condition throughout, which guarantees that $L_{\mathcal{I}} \neq \varnothing$ for all $\mathcal{I} \subset \mathcal{K}$.

> **Principle Identification**: $X_0 \neq \varnothing$, and, $X_k \backslash \cup_{k' \neq k} X_{k'} \neq \varnothing$ for every $k \in \mathcal{K}$.

Under this condition, $L_{\mathcal{I}}$ is nonempty and consists of lotteries that only assign positive probabilities to $X_0$ and $X_k \backslash \cup_{k' \notin \mathcal{I}} X_{k'}$ for all $k \in \mathcal{I}$.

As before, we consider a binary relation $\succsim$ on $\Delta(X)$, referred to as the preference. We will make use of the following notions, that adapt the previous definition of overlapping preference to the multiple-principle setup:

**Definition 5.** *We say that*

- *for every $I, I' \subset \mathcal{K}$, $O_{\mathcal{I}\mathcal{I}'} = \{(P,Q) : P \sim Q, P \in L_{\mathcal{I}}, Q \in L_{\mathcal{I}'}\}$ denotes the preference overlapping between $L_{\mathcal{I}}$ and $L_{\mathcal{I}'}$;*
- *$L_{\mathcal{I}}$ and $L_{\mathcal{I}'}$ are* $\succsim$-connected, *or, simply,* connected *if $O_{\mathcal{I}\mathcal{I}'} \neq \varnothing$;*
- *for any set $\mathbb{S} \subset 2^{\mathcal{K}}$, $\{L_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}}$ is connected if, for every $I, I' \in \mathbb{S}$, there exists a finite sequence $\{L_{\mathcal{I}(n)}\}_{n=1}^{N}$ in $\{L_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}}$ with $L_{\mathcal{I}(1)} = L_{\mathcal{I}}$, $L_{\mathcal{I}(N)} = L_{\mathcal{I}'}$, and $L_{\mathcal{I}(n)}$ and $L_{\mathcal{I}(n+1)}$ are connected, for $1 \leq n \leq N-1$.*

We now present two multiple-principle analogies to model (2.2). The general version associates every subset of principles, $\mathcal{I}$, with a real-number $b_{\mathcal{I}}$, which measures the cost of violating (all and only) the principles in $\mathcal{I}$. The special version presents the refinement that every $b_{\mathcal{I}}$ is equal to the sum of the costs of individual principles in $\mathcal{I}$, that is $b_{\mathcal{I}} = \sum_{k\in\mathcal{I}} b_k$.

### 3.2 The General Case

We start by adapting a few axioms used for the single-principle case to the more general multiple-principle version. Our exposition here will be concise as the content of these properties has been already discussed in detail for the single-principle case. The first axiom, generalizing Semi-Restricted Continuity (A2*), maintains continuity when mixing lotteries that belong to the same $L_{\mathcal{I}}$, because the mixture will still violate the principles in $\mathcal{I}$ and, hence, it is in $L_{\mathcal{I}}$, as well.

> **MP 1. Semi-Restricted Continuity:** For every $\mathcal{I} \subset \mathcal{K}$, $P, R \in L_{\mathcal{I}}$, and $Q \in L$, if $P \succ Q \succ R$, then there exist $\alpha, \beta \in (0,1)$ such that
>
> $$\alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R.$$

Note that Axiom MP1 reduces to Axiom A2* in the special case of one single principle, i.e., if $\mathcal{I} = \{1\}$. The second axiom preserves independence within each $L_{\mathcal{I}}$, and, in addition to that, it applies to the case in which two lotteries in some $L_{\mathcal{I}}$ are mixed with a lottery in a different $L_{\mathcal{K}}$. The idea is that mixing with some $R \in L_{\mathcal{K}}$ brings in further violations of the principles in $\mathcal{K}\backslash\mathcal{I}$ to both mixture lotteries, which are therefore expected to be penalized by the same amount. This is a direct generalization of Semi-Restricted Independence (A3*): indeed, the original formulation of A3* is recovered by setting $\mathcal{K} = \{1\}$ and $\mathcal{I} = \emptyset$.

**MP 2. Semi-Restricted Independence:** For every $\mathcal{I} \subset \mathcal{K}$, $P, Q \in L_{\mathcal{I}}$, $R \in L_{\mathcal{I}} \cup L_{\mathcal{K}}$, and $\alpha \in (0,1)$,

$$P \succsim Q \Longleftrightarrow \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R.$$

The third axiom concerns comparisons across any two different spaces $L_{\mathcal{I}}$ and $L_{\mathcal{I}'}$, and it is a direct generalization of $L_0$-Cancellation (A6). Indeed, if we have only one principle, then $\mathcal{K} = \mathcal{I} = \{1\}, \mathcal{I}' = \{\emptyset\}$, and Axiom MP3 coincides with the Axiom A6. The original content of this property is thus intact: a commonly shared zero-risk component should not affect the ranking of any two lotteries.

**MP 3. $L_{\varnothing}$-Cancellation:** For every $\mathcal{I}, \mathcal{I}' \subset \mathcal{K}$, $P \in L_{\mathcal{I}}$, $Q \in L_{\mathcal{I}'}$, $R, R' \in L_{\varnothing}$, and $\alpha \in (0,1)$,

$$\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)R \implies \alpha P + (1-\alpha)R' \sim \alpha Q + (1-\alpha)R'.$$

**Theorem 2.** *Assume that Principle Identification holds, and that the preference $\succsim$ on $L$ is a weak order such that there exist $S, T \in L_{\varnothing}$ with $S \succ T$. Then, the following statements are equivalent:*

*(i) The preference $\succsim$ satisfies MP1-MP3;*

*(ii) There exist an affine utility function $U$ on $L$, denoting the expected utility for some utility function $u : X \to \mathbb{R}$ that is not constant on $X_0$, and a set of real numbers $\{b_{\mathcal{I}}\}_{\mathcal{I} \subset K}$ such that, for every $\mathcal{I}, \mathcal{I}' \subset \mathcal{K}$, $P \in L_{\mathcal{I}}$, and $Q \in L_{\mathcal{I}'}$,*

$$P \succsim Q \Longleftrightarrow U(P) - b_{\mathcal{I}} \geq U(Q) - b_{\mathcal{I}'}.$$

*Moreover, $u$ is unique up to affine transformations, and, for every $I, I' \subset \mathcal{K}$, if $L_{\mathcal{I}}$ and $L_{\mathcal{I}'}$ connected, then $b_{\mathcal{I}} - b_{\mathcal{I}'}$ is unique given $u$. In particular, $\{b_{\mathcal{I}}\}_{\mathcal{I} \subset K}$ is unique up to translations given $u$ if $\{L_{\mathcal{I}}\}_{\mathcal{I} \subset \mathcal{K}}$ is connected.*

Theorem 2 is a direct generalization of Theorem 1 to the case of multiple principles. The overall assessment of a lottery $P \in L_{\mathcal{I}}$ is now determined by the sum of its hedonic value – measured, as usual, by the utility $U$ – and the cost associated to the set $\mathcal{I}$ of principles that risk being violated – measured by the parameter $b_{\mathcal{I}}$. In particular, for every possible set $\mathcal{I}$ of principles, the above result identifies uniquely a penalty $b_{\mathcal{I}}$ arising from accepting the risk of violating all, and only, the principles in the set $\mathcal{I}$. Note that Theorem 2 accommodates several types of comparisons. In particular, it allows to compare not only safe lotteries with lotteries that violate a certain set of principles, but also to compare risky lotteries that violate different sets of principles. Furthermore, it also provides guidance in the comparisons of two lotteries that violate a single but distinct principle: for instance, suppose that an agent considers two principles, given by $\mathcal{K} = \{1, 2\}$, and compares a lottery $P \in L_1$, violating principle 1, with a lottery $Q \in L_2$, violating principle 2. The two principles can be associated to different costs, thereby making the comparison generally nontrivial. To illustrate, we can turn back to Example 2 discussed in the Introduction about atomic weapons. In that historical background, one could think of two principles at stake: the principle of not risking the Nazi to win World War II and the principle of not risking to destroy the world with the atomic bomb. Abiding by either of these two principles would entail choosing an action (respectively, developing atomic weapons or refraining from doing it) which would be in conflict with the other principle. Our model provides a formal framework to reason about difficult problems involving principles in conflict with each others, and it indicates that the decision process should be based on the relative comparisons of material well-being and costs associated to each action.

Finally, note that in Theorem 2, the cost $b_{\mathcal{I}}$ for all $\mathcal{I} \subset \mathcal{K}$ can be any real number. However, it is clear that we can adopt similar conditions as the Positivity condition of Section 2.3 to make the costs strictly positive. And, infinite outcomes can be accommodated by an extra condition in the spirit of the Archimedeanity of Section 2.4.

We next give a brief summary of the proof.

**Proof sketch.** To start, note that MP1 and MP2 imply the standard vNM axioms on each $L_{\mathcal{I}}$ and, hence, we obtain vNM expected utility functions $\{U_{\mathcal{I}}\}_{\mathcal{I} \subset \mathcal{K}}$. We then show that there exists a single expected utility function $U$ defined on the entire domain $L = \Delta(X)$, and for any two $\mathcal{I}, \mathcal{I}' \subset \mathcal{K}$, there exists a number $b_{\mathcal{I}\mathcal{I}'} \in \mathbb{R}$ such that $U + b_{\mathcal{I}\mathcal{I}'}$ on $L_{\mathcal{I}}$ and $U$ on $L_{\mathcal{I}'}$ jointly represent $\succsim$ on $L_{\mathcal{I}} \cup L_{\mathcal{I}'}$. This is mostly due to MP3 when $O_{\mathcal{I}\mathcal{I}'}$ is nontrivial (that is, there exist $(P, P')$ and $(Q, Q')$ in $O_{\mathcal{I}\mathcal{I}'}$ with $P \succ Q$); and, it is due to MP2 otherwise, by arguments similar to those in Proposition 2 and 4.[8] Now we need to reduce the number of parameters in $\{b_{\mathcal{I}\mathcal{I}'}\}_{\mathcal{I}, \mathcal{I}' \subset \mathcal{K}}$ to that of $\{b_{\mathcal{I}}\}_{\mathcal{I} \subset \mathcal{K}}$, and a main message of the proof is that this reduction comes for free. The key step entails showing that, for example, for any loop of $L_{\mathcal{I}}$'s, by which we mean a finite collection $\{L_{\mathcal{I}(n)}\}_{n=1}^{N}$ with $L_{\mathcal{I}(1)} = L_{\mathcal{I}(N)}$ and $L_{\mathcal{I}(n)}$ being connected to $L_{\mathcal{I}(n+1)}$ for all $1 \leq n \leq N-1$, it must be that $\sum_{n=1}^{N-1} b_{\mathcal{I}(n)\mathcal{I}(n+1)} = 0$. Here the proof essentially relies on induction. Once established, this kind of property allows us to obtain the desired result for any connected collection of $L_{\mathcal{I}}$'s. Finally, we divide $\{L_{\mathcal{I}}\}_{\mathcal{I} \subset \mathcal{K}}$ into disjoint (in preference terms) sub families of connected $L_{\mathcal{I}}$'s. Representation results on each of the sub families can be brought together to deliver an unified model.

[8] For this step, we could have opted for a generalization of A5 (that is, replacing $L_0$ and $L_1$ by any two $L_{\mathcal{I}}$ and $L_{\mathcal{I}'}$) as opposed to MP 3, which is a generalization of A6. However, mixing $P \in L_{\mathcal{I}}$ with $Q \in L_{\mathcal{I}'}$, as needed by a generalization of A5, will result in a lottery that is in $L_{\mathcal{I} \cup \mathcal{I}'}$, hence bringing in a third domain $L_{\mathcal{I} \cup \mathcal{I}'}$ to the analysis. MP 3, on the other hand, allows us to stay in the two domains $L_{\mathcal{I}}$ and $L_{\mathcal{I}'}$.

### 3.3 The Additive Case

A special case of Theorem 2 associates every principle $k \in \mathcal{K}$ with a number $b_k$, conveying the idea that risking principle $k$ induces cost $b_k$ and risking a set $\mathcal{I} \subset \mathcal{K}$ of principles induces cost $\sum_{k\in\mathcal{I}} b_k$. Here the trade-off between expected (hedonic) utility and the principles is treated in an additive manner, and, on top of that, the cost of violating a set of principles is also additive in the individual costs. We now investigate the preference condition(s) for this extra additivity in costs.

For example, consider the two-principle case with $\mathcal{K} = \{1, 2\}$. Suppose that there exist $P_1 \in L_\varnothing$, $Q_1 \in L_{\{1\}}$, $P_2 \in L_{\{1,2\}}$ and $Q_2 \in L_{\{2\}}$ with $P_1 \sim Q_1$ and $P_2 \sim Q_2$. Additive cost entails that the (hedonic) utility difference between $Q_1$ and $P_1$ and that between $P_2$ and $Q_2$ should be the same, as they are both equal to the cost of violating principle 1. In preference terms, this can be stated as $\frac{1}{2}P_1 + \frac{1}{2}P_2 \sim \frac{1}{2}Q_1 + \frac{1}{2}Q_2$. Similarly, if there exist $P_1, P_2, Q_3 \in L_\varnothing$, $Q_1 \in L_{\{1\}}$, $Q_2 \in L_{\{2\}}$, and $P_3 \in L_{\{1,2\}}$ with $P_1 \sim Q_1$, $P_2 \sim Q_2$, $P_3 \sim Q_3$, additive cost mandates that the utility difference between $P_3$ and $Q_3$, which measures the cost of violating both principles, should be equal to the sum of the utility difference between $Q_1$ and $P_1$ and that between $Q_2$ and $P_2$, which measure the cost of the two principles separately. This can be stated in preference terms as $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3 \sim \frac{1}{3}Q_1 + \frac{1}{3}Q_2 + \frac{1}{3}Q_3$.

The general idea can be formulated as follows. Notice that the symbol $1_\mathcal{I}$ for any $\mathcal{I} \subset \mathcal{K}$ stands for the indicator function on $\mathcal{K}$ that takes value 1 on $\mathcal{I}$ and 0 on $\mathcal{I}^c$.

> **Additivity**: For any finite collection of $P_n \in L_{\mathcal{I}_n}$ and $Q_n \in L_{\mathcal{J}_n}$ with $P_n \sim Q_n$, $1 \leq n \leq N$, if $\sum_{n=1}^N 1_{\mathcal{I}_n} = \sum_{n=1}^N 1_{\mathcal{J}_n}$, then $\sum_{n=1}^N \frac{1}{N} P_n \sim \sum_{n=1}^N \frac{1}{N} Q_n$.[9]

The next result shows that Additivity delivers a representation where the overall cost of violating a given set of principles is given by the sum of the costs associated with each principle in the set.

**Proposition 8.** *Assume that the conditions and the axioms of Theorem 2 hold, and that $\{L_\mathcal{I}\}_{\mathcal{I}\subset\mathcal{K}}$ is connected. Additivity holds if and only if there exist an affine utility function $U$ on $L$ and a set of real numbers $\{b_k\}_{k\in\mathcal{K}}$ such that, for every $P \in L_\mathcal{I}$ and $Q \in L_{\mathcal{I}'}$ with $\mathcal{I}, \mathcal{I}' \subset \mathcal{K}$,*

$$P \succsim Q \Longleftrightarrow U(P) - \sum_{k\in\mathcal{I}} b_k \geq U(Q) - \sum_{k\in\mathcal{I}'} b_k. \tag{3.1}$$

*The function $U$ is unique up to affine transformations, and, the set $\{b_k\}_{k\in\mathcal{K}}$ is unique given $U$.*

### 3.4 Discussion of the Results

Theorem 2 associates a single cost $b_\mathcal{I}$ for choosing lotteries that violate a set $\mathcal{I}$ of principles. It is silent about how this overall measure comprises the costs associated to each of the

[9] See Kraft, Pratt, and Seidenberg (1959), who also used a similar condition, based on summations of indicator functions, to obtain an additive structure.

principles in $\mathcal{I}$. Proposition 8 adds important structural specifications by allowing us to map the violation of each principle $k$ in $\mathcal{I}$ to a corresponding cost $b_k$ for the agent. It therefore allows not only to identify the sets of principles that are relevant for an agent, but also to elicit the relative importance of the principles within a set. This extra structure substantially enlarges the range of applicability of our model. Indeed, similarly to the more general Theorem 2, this result can be used to address problems where the agent holds a set of principles that are potentially in conflict with each other. While Theorem 2 does not restrict the way different principles are aggregated to make decisions, Proposition 8 imposes a linear structure and maintains that the total cost is given by the sum of the individual costs associated to each violated principle. We can interpret $\{b_k\}_{k\in\mathcal{K}}$ as identifying an individual's system of values. Representation (3.1) then captures the behavior of a decision maker who entertains a preference for "zero risk" for multiple types of risk. When a conflict among different values arises, our representation can help to guide decisions by suggesting that the conflict resolution will hinge on the way the relative importance of the individual's principles contributes to the overall utility together with material considerations. For instance, one may hold two principles, the one of not risking a child's life and the other of not risking her mental health; and, then, she may face the dilemma of choosing to live in a less-safe country where her child feels at home versus choosing a safe country in which she would be an outsider.

Finally, Proposition 8 can be used to compare agents in terms of the relevance they attribute to each principle. In particular, our model allows to estimate the effects of each value on choice. From an applied perspective, such estimations can be informative to public institutions and can be used to elicit citizens' support for a certain policy. For instance, many countries face difficult decisions regarding nuclear energy, especially in light of climate change. Nuclear energy is nonpollutant as long as all goes well, but it can obviously be very dangerous. Is there any level of safety of nuclear plants that would make them an acceptable source of energy? Expected utility theory suggests that the answer is in the affirmative. Yet, many seem to think differently, and to view any positive level of risk as different from zero. Further, in many countries this seems to be the dominant public view and sometimes also the implemented government policy.

To sum, our representations provide modelling tools that: (i) allow to formally describe the modes of actual behavior that arise from stating discontinuous preferences; (ii) normatively guide individuals' reasoning process when the decision process is driven by "zero-risk" principles; and (iii) arm an external observer (say, a policy maker or a public institution) with the necessary toolkit to estimate the importance of a certain principle within a society and, therefore, to elicit how much material well-being a society is willing to sacrifice to abide by that principle. This is an important step to assess the support of a society for a certain policy. Admittedly, the standard paradigm of expected utility does not seem capable of addressing all these issues.

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# 5 Appendix: Proofs and Related Analysis

## 5.1 Proof of Proposition 1

**Lemma 1.** *Axioms A1-A3 hold if and only if there exist $u_0$ on $X_0$ and $u_1$ on $X$ such that $U_i$ represents $\succsim$ on $L_i$ for $i \in \{0,1\}$.*

*Proof.* The sets $L_0$ and $L_1$ are mixture sets in the sense of Herstein and Milnor (1953). The vNM axioms that we impose on $L_0$ and $L_1$, separately, are simply the Herstein and Milnor axioms applied to lottery mixtures. The Herstein and Milnor result is therefore directly applicable, and we get $u_0$ on $X_0$ and $u_1$ on $X$ such that $U_i$ represents $\succsim$ on $L_i$, $i \in \{0,1\}$. $\square$

From now on we shall take Axioms A1-A3 as implicitly given and refer to $u_i$ and $U_i$ freely, $i \in \{0,1\}$.

**Lemma 2.** *(Solvability) Under A2*, for every $P, Q, R \in L$ with $P \succ Q \succ R$, if $P, R \in L_0$ or $P, R \in L_1$, then there exists $\beta \in (0,1)$ such that $\beta P + (1-\beta)R \sim Q$.*

*Proof.* The sets $B_+ = \{\beta \in (0,1) : \beta P + (1-\beta)R \succ Q\}$ and $B_- = \{\beta \in (0,1) : Q \succ \beta P + (1-\beta)R\}$ are nonempty and open by A2* and are disjoint. Then the set $B_\sim = (0,1)\backslash(B_+ \cup B_-)$ has to be nonempty by the connectedness of $(0,1)$ and indeed a singleton by A3. Take the $\beta^* \in B_\sim$, it must be that $\beta^* P + (1-\beta^*)R \sim Q$. $\square$

We can now establish Proposition 1.

*Proof.* We focus on A2*'s sufficiency, as its necessity is straightforward.

Both $O_{0,\sim}$ and $O_{1,\sim}$ are convex. For all $P, R \in O_{0,\sim}$, let $P', R' \in O_{1,\sim}$ be lotteries such that $P \sim P'$ and $R \sim R'$. If $P \sim R$, then for all $\alpha \in (0,1)$, $\alpha P + (1-\alpha)R \sim P \sim P'$ and, therefore, $\alpha P + (1-\alpha)R \in O_{0,\sim}$. If $P \succ R$, then for all $\alpha \in (0,1)$, $P' \succ \alpha P + (1-\alpha)R \succ R'$. By the solvability property, there exists a $\beta \in (0,1)$ with $\beta P' + (1-\beta)R' \sim \alpha P + (1-\alpha)R$, implying that $\alpha P + (1-\alpha)R \in O_{0,\sim}$. The case for $P, R \in O_{1,\sim}$ is similar.

For all $R \in L_0\backslash O_{0,\sim}$, either $R \succ L_1$ or $L_1 \succ R$. This follows directly from the solvability property, because if there are $R \in L_0\backslash O_{0,\sim}$ and $P, Q \in L_1$ with $P \succsim R \succsim Q$, $R$ must be in

$O_{0,\sim}$, a contradiction. Likewise, for all $R \in L_1 \backslash O_{1,\sim}$, either $R \succ L_0$ or $L_0 \succ R$. It follows immediately that if $P \in L_0 \backslash O_{0,\sim}$ and $P \succ L_1$, then we cannot have $Q \in L_1 \backslash O_{1,\sim}$ with $Q \succ L_0$, and similar statements can be made for the other three scenarios.

If $O_\sim = \emptyset$, from the last paragraph we must have either $L_0 \succ L_1$ or $L_1 \succ L_0$. In the first case, let $g(U_0(P)) = (sup_{S \in L_1} U_1(S) + 1) + (U_0(P) - inf_{S \in L_0} U_0(S))$ for all $P \in L_0$. And, in the latter case, let $g(U_0(P)) = (inf_{S \in L_1} U_1(S) - 1) + (U_0(P) - sup_{S \in L_0} U_0(S))$ for all $P \in L_0$. This $g$ function simply translates $U_0$ by a constant and hence it is strictly increasing and continuous. By construction, along with $U_1$, it represent $\succsim$.

From now on we assume $O_\sim \neq \emptyset$. The function $g$ can be constructed as follows. For every $R \in O_{0,\sim}$, take any $R' \in L_1$ with $R' \sim R$ and let $g(U_0(R)) = U_1(R')$. For any $R \in L_0 \backslash O_{0,\sim}$ with $R \succ L_1$, let $g(U_0(R)) = \big(U_0(R) - sup_{S \in O_{0,\sim}} U_0(S)\big) + sup_{S \in O_{0,\sim}} g(U_0(S))$. For any $R \in L_0 \backslash O_{0,\sim}$, with $L_1 \succ R$, let $g(U_0(R)) = \big(U_0(R) - inf_{S \in O_{0,\sim}} U_o(S)\big) + inf_{S \in O_{0,\sim}} g(U_0(S))$.

We now prove that function $g$ is continuous. Let us first analyze $g$ restricted to the convex interval $U_0(O_{0,\sim}) \subset \mathbb{R}$. Since $g$ is strictly increasing, discontinuity takes the form that for some lottery $P \in O_{0,\sim}$, either the left limit of $g$ at $U_0(P)$, $g_-(U_0(P))$, is strictly smaller than $g(U_0(P))$, or the right limit at $U_0(P)$, $g_+(U_0(P))$, is strictly larger than $g(U_0(P))$. In the former case, take any $Q \in O_{0,\sim}$ with $U_0(Q) < U_0(P)$ and let $P', Q' \in O_{1,\sim}$ be the lotteries such that $P \sim P'$ and $Q \sim Q'$. By definition we have $g(U_0(P)) = U_1(P') > g(U_0(Q)) = U_1(Q')$ and there exists $\beta \in (0,1)$ such that $U_1(\beta P' + (1-\beta)Q') \in (g_-(U_0(P)), g(U_0(P)))$. This indicates that $\beta P' + (1-\beta)Q' \notin O_{1,\sim}$, which contradicts the convexity of $O_{1,\sim}$. The argument that rules out $g_+(U_0(P)) > g(U_0(P))$ is similar. On $U_0(L_0) \backslash U_0(O_{0,\sim})$ and the boundary of $U_0(O_{0,\sim})$, $g$ is by construction linear and hence continuous.

It remains to prove that the functions $g \circ U_0$ and $U_1$ together represent $\succsim$. The statement is obviously true for any $P, Q$ that are both in $L_0$ or $L_1$, since the function $g$ is strictly increasing by construction. Thus, consider the case where $P \in L_0$ and $Q \in L_1$, and assume that $g(U_0(P)) \geq U_1(Q)$. If $P \in O_{0,\sim}$ and hence there exists $P' \in L_1$ with $P' \sim P$, we then have $U_1(P') = g(U_0(P)) \geq U_1(Q)$ and therefore $P \sim P' \succsim Q$. If $P \in L_0 \backslash O_{0,\sim}$ but $L_1 \succ P$, then either $Q \in O_{1,\sim}$ or $Q \succ O_{1,\sim}$, which implies $Q \succsim S$ for some $S \in O_{1,\sim}$. For any $S' \in L_0$ with $S' \sim S$, we then have $Q \succsim S \sim S' \succ P$ and hence $U_1(Q) \geq U_1(S) = g(U_0(S')) > g(U_0(P))$, a contradiction. So, it must be that $P \succ L_1$ and $P \succ Q$ in particular. The case of $g(U_0(P)) \leq U_1(Q)$ implying $Q \succsim P$ follows from similar arguments. Vice versa, assume that $P \succsim Q$ with $P \in L_0$ and $Q \in L_1$. If $P \in O_{0,\sim}$, there exists $P' \in O_{1,\sim}$ with $P' \sim P \succsim Q$, from which we get $g(U_0(P)) = U_1(P') \geq U_1(Q)$. If $P \in L_0 \backslash O_{0,\sim}$, then $P \succ L_1$ and there exists $R \in O_{0,\sim}$ with $P \succ R \succsim Q$, from which we get $g(U_0(P)) > g(U_0(R)) \geq U_1(Q)$. The case for $P \succsim Q$ with $P \in L_1$ and $Q \in L_0$ can be similarly addressed. □

## 5.2 Proof of Proposition 2

*Proof.* For both statements we shall focus on the sufficiency of the axioms, as their necessity is straightforward.

(1) For all $P, Q \in L_0$ and $R \in L_1$, we have $P \succ Q$ if and only if $U_1(\alpha P + (1-\alpha)R) > U_1(\alpha Q + (1-\alpha)R)$. Since $U_1$ is linear, we have $P \succ Q$ if and only if $U_1(P) > U_1(Q)$, which implies that $U_0$ and $U_1$ are cardinally equivalent.

(2) If $O_{0,\sim} = \emptyset$ or $U_0(O_{0,\sim})$ is a singleton, the function $g \circ U_0$ is, by construction, a translation of $U_0$ and, hence, $g(\cdot)$ is affine. We next assume that $U_0(O_{0,\sim})$ is a non-degenerate interval. A4 implies that for any $P, R \in O_{0,\sim}$ and $P', R' \in O_{1,\sim}$ with $P \sim P' \succ R \sim R'$, we have $g(U_0(\frac{1}{2}P + \frac{1}{2}R)) = U_1(\frac{1}{2}P' + \frac{1}{2}R')$. Since $g(U_0(P)) = U_1(P')$ and $g(U_0(R)) = U_1(R')$ by construction, we obtain, by the linearity of $U_0$ and $U_1$, that $g(\frac{1}{2}U_0(P) + \frac{1}{2}U_0(R)) = \frac{1}{2}g(U_0(P)) + \frac{1}{2}g(U_0(R))$. Since $P, R \in O_{0,\sim}$ are arbitrary and $g$ is continuous, we conclude that $g$ is affine on $U(O_{0,\sim})$. That is, there are a scaling parameter $a > 0$ and a translation parameter $b \in \mathbb{R}$ such that $g(x) = ax + b$ for all $x \in U(O_{0,\sim})$.

For any $R \in L_0 \backslash O_{0,\sim}$ with $R \succ L_1$, let

$$g(U_0(R)) = a\left(U_0(R) - sup_{S \in O_{0,\sim}} U_o(S)\right) + sup_{S \in O_{0,\sim}} g(U_0(S)).$$

For any $R \in L_0 \backslash O_{0,\sim}$, with $L_1 \succ R$, let

$$g(U_0(R)) = a\left(U_0(R) - inf_{S \in O_{0,\sim}} U_o(S)\right) + inf_{S \in O_{0,\sim}} g(U_0(S)).$$

Function $g(\cdot)$ is thus extended to $U_0(L_0)$ in its entirety and is still affine with the same scaling parameter $a > 0$ and translation parameter $b$.

□

### 5.3 Proof of Proposition 3

*Proof.* We first show that A5 implies A3*. Notice that when $P, Q, R \in L_0$ or $P, Q \in L_1$, A3* coincides with A3. So we shall focus on the case $P, Q \in L_0$ and $R \in L_1$. Suppose $P, Q \in O_{0,\sim}$ and take $P', Q' \in O_{1,\sim}$ with $P \sim P' \succsim Q \sim Q'$. We have

$$\frac{1}{2}P' + \frac{1}{2}P \succsim \frac{1}{2}Q' + \frac{1}{2}P \sim \frac{1}{2}P' + \frac{1}{2}Q,$$

where the first part is by the linearity of $\succsim$ on $L_1$ and the indifference part is by A5. Therefore we have $\frac{1}{2}P + \frac{1}{2}P' \succsim \frac{1}{2}Q + \frac{1}{2}P'$. By the linearity of $\succsim$ on $L_1$ again, we get $\alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$ for all $a \in (0,1)$ and $R \in L_1$.

For general $P, Q \in L_0$ with $P \succsim Q$, fix $S \in O_{0,\sim}$ with $U_0(S)$ in the interior of the interval $U_0(O_{0,\sim})$. There exists $\beta \in (0,1)$ such that $\beta P + (1-\beta)S$ and $\beta Q + (1-\beta)S$ are in $O_{0,\sim}$ and $\beta P + (1-\beta)S \succsim \beta Q + (1-\beta)S$ by the linearity of $\succsim$ on $L_0$. Now we can use the argument in the previous paragraph to conclude that, for all $\alpha \in (0,1)$ and $R \in L_1$,

$\alpha(\beta P + (1-\beta)S) + (1-\alpha)R \succsim \alpha(\beta Q + (1-\beta)S) + (1-\alpha)R$, which can be re-written as

$$\alpha\beta P + (1-\alpha\beta)\left(\frac{\alpha-\alpha\beta}{1-\alpha\beta}S + \frac{1-\alpha}{1-\alpha\beta}R\right) \succsim \alpha\beta Q + (1-\alpha\beta)\left(\frac{\alpha-\alpha\beta}{1-\alpha\beta}S + \frac{1-\alpha}{1-\alpha\beta}R\right).$$

As $\frac{\alpha-\alpha\beta}{1-\alpha\beta}S + \frac{1-\alpha}{1-\alpha\beta}R \in L_1$, we have by the linearity of $\succsim$ on $L_1$ that $\gamma P + (1-\gamma)R \succsim \gamma Q + (1-\gamma)R$ for all $\gamma \in (0,1)$ and $R \in L_1$.

Notice that, for the last two paragraphs, the conclusion will be indifference if $P \sim Q$ and strict preference if $P \succ Q$. A3* is thus established.

It remains to show that A5 implies A4. Take any $P, Q \in O_{0,\sim}$ and $P', Q' \in O_{1,\sim}$ with $P \sim P' \succsim Q \sim Q'$. In the case of $P \sim Q$, we have $\frac{1}{2}P + \frac{1}{2}Q \sim P \sim P' \sim \frac{1}{2}P' + \frac{1}{2}Q'$. If $P \succ Q$, by A2* there exists a unique $\beta \in (0,1)$ such that $\frac{1}{2}P + \frac{1}{2}Q \sim \beta P' + (1-\beta)Q'$. By A5 we have $\frac{1}{2}P + \frac{1}{2}Q' \sim \frac{1}{2}P' + \frac{1}{2}Q$ and also $\frac{1}{2}(\frac{1}{2}P + \frac{1}{2}Q) + \frac{1}{2}Q' \sim \frac{1}{2}(\beta P' + (1-\beta)Q') + \frac{1}{2}Q$, which by the linearity of $\succsim$ on $L_1$ imply that $U_1(P) - U_1(Q) = U_1(P') - U_1(Q')$ and $U_1(P) - U_1(Q) = 2\beta(U_1(P') - U_1(Q'))$, respectively. Hence $\beta$ can only be $\frac{1}{2}$. $\square$

### 5.4 Proof of Theorem 1

*Proof.* We focus on the sufficiency of the axioms, as their necessity is straightforward.

If $O_\sim$ is not nontrivial, then $U_0(O_{0,\sim})$ is either empty or a singleton, and, by A3* and the argument in part (1) of the proof of Proposition 2, we take $U_0$ to be $U_1$ restricted on $L_0$. If $U(O_{0,\sim}) = \emptyset$ and $L_0 \succ L_1$, take any $b > supU(L_1) - infU(L_1)$. If $U_0(O_{0,\sim}) = \emptyset$ and $L_1 \succ L_0$ take any $b < infU(L_1) - supU(L_1)$. If $U_0(O_{0,\sim})$ is a singleton, let $b = U_1(P') - U_1(P)$, with $(P, P') \in O_\sim$.

Suppose that $O_\sim$ is nontrivial. A5 means that, for all $P, R \in L_0$ and $P', R' \in L_1$ with $P \sim P' \succ R \sim R'$, $U_1(\frac{1}{2}P + \frac{1}{2}R') = U_1(\frac{1}{2}P' + \frac{1}{2}R)$. Since $g(U_0(P)) = U_1(P')$ and $g(U_0(R)) = U_1(R')$, it follows that $\frac{1}{2}U_1(P) + \frac{1}{2}g(U_0(R)) = \frac{1}{2}g(U_0(P)) + \frac{1}{2}U_1(R)$ and, hence, $g(U_0(P)) - g(U_0(R)) = U_1(P) - U_1(R)$ for all $P, R \in O_{0,\sim}$. Thus, there exists $b \in \mathbb{R}$ such that $g(U_0(P)) = U_1(P) + b$ for all $P \in O_{0,\sim}$. In particular, $U_1$ represents $\succsim$ on $O_{0,\sim}$.

Since $\{P \in L_0 : P \in O_{0,\sim}\}$ contains an open set, $U_1$ and $U_0$ must be cardinally the same on $L_0$. We can hence take $U_0$ to be $U_1$ restricted on $L_0$. And, for any $R \in L_0 \backslash O_{0,\sim}$ let $g(U_0(R)) = U_0(R) + b = U_1(R) + b$. $\square$

### 5.5 Proof of Proposition 4

*Proof.* Let $P_u \in L_0$ denote the uniform distribution over $X_0$. Notice that for any number $a$ in the interior of $U_0(O_{0,\sim})$ and any number $b$ in the interior of $U_1(O_{1,\sim})$, there exist $P \in L_0$, $Q \in L_1$, and $\alpha, \beta \in (0,1)$ such that $U_0(\alpha P_u + (1-\alpha)P) = a$ and $U_1(\beta P_u + (1-\beta)Q) = b$. This is true because we can pick any $P \in L_0$ with $U_0(P) < a$ if $U_0(P_u) > a$, any $P \in L_0$ with $U_0(P) > a$ if $U_0(P_u) < a$, and $P = P_u$ if $U_0(P_u) = a$. Similar arguments apply for $L_1$ except for $U_1(P_u) = b$, in which case we just take any $Q \in L_1$ with $U_1(Q) = b$. Since

$\alpha P_u + (1-\alpha)P = \alpha' P_u + (1-\alpha')(\frac{1-\alpha}{1-\alpha'}P + \frac{\alpha-\alpha'}{1-\alpha'}P_u)$ for all $\alpha' \in (0,\alpha)$ and a similar property holds for $\beta' \in (0,\beta)$, we can assume that $\alpha = \beta$ without loss of generality.

By the above claim there exist $P \in L_0$, $Q \in L_1$, and $\alpha \in (0,1)$ such that $U(\alpha P_u + (1-\alpha)P)$ is in the interior of $U_0(O_{0,\sim})$ and $\alpha P_u + (1-\alpha)P \sim \alpha P_u + (1-\alpha)Q$. Therefore by A6 we have $\alpha R + (1-\alpha)P \sim \alpha R + (1-\alpha)Q$ for all $R \in L_0$. In particular, we have

$$\alpha R + (1-\alpha)P \succsim \alpha R' + (1-\alpha)P \Longleftrightarrow \alpha R + (1-\alpha)Q \succsim \alpha R' + (1-\alpha)Q,$$

as the first lottery and the third lottery are indifferent and so are the second and the fourth. Under intra-space independence, this implies that $U_0$ and $U_1$ are cardinally equivalent on $L_0$. This property obviously implies A3*.

For the rest of the proof, we use function $U$ to denote both $U_0$ and $U_1$. For A4 and A5, we shall show that the utility transformation $g(\cdot)$ between $U$ on $L_0$ and $U$ on $L_1$ is a translation. The idea is to show that for each $P \in O_{0,\sim}$ with $U(P)$ in the interior of $U(O_{0,\sim})$, $g$ at a neighborhood of $U(P)$ is locally a translation. We then glue this local property together to show that $g$ is a translation on the whole domain $U(O_{0,\sim})$.

For any number $a$ in the interior of $U(O_{0,\sim})$, take $\alpha_a P_u + (1-\alpha_a)P \in L_0$, $\alpha_a P_u + (1-\alpha_a)Q \in L_1$, and $\alpha_a \in (0,1)$ with $U(\alpha_a P_u + (1-\alpha_a)P) = a$ and $U(\alpha_a P_u + (1-\alpha_a)Q) = g(a)$. Let $H_a$ denote the convex interval $\alpha_a\,(U(L_0) - U(P_u))$ excluding its boundary, which is therefore open. By A6, we can conclude that

$$g(a+b) = g(a) + b, \forall b \in H_a.$$

That is, $g$ is locally a translation of $a + H_a$ to $g(a) + H_a$. Note that $U(O_{0,\sim})$ excluding its boundary can be expressed as $\cup_{n=1}^{\infty} C_n$, where each $C_n$ is a compact convex intervals and $C_n \subset C_{n+1}$ for all $n \geq 1$. Each $C_n$ is covered by the open intervals $\{a + H_a\}_{a \in C_n}$ and hence there exists a finite sub-cover $\{a_m + H_{a_m}\}_{m=1}^{M_n}$ with $a_m < a_{m+1}$ for all $1 \leq m \leq M_n - 1$. Since in this sub-cover any $a_m + H_{a_m}$ has a nonempty intersection with some other $a_{m'} + H_{a_{m'}}$, and the intersection itself is an open interval, it must be that $g$ is a translation on $(a_m + H_{a_m}) \cup (a_{m'} + H_{a_{m'}})$. By induction on $m$, we can conclude that $g$ is a translation on $C_n$, and hence also on $\cup_{n=1}^{\infty} C_n$. Since $g$ is continuous on $U(O_{0,\sim})$, we can conclude that $g$ is a translation on $U(O_{0,\sim})$. By construction we can also make $g$ a translation on the whole of $U(L_0)$. $\square$

### 5.6 Proof of Proposition 5

*Proof.* Since $O_\sim$ is nonempty, we have $P \sim Q$ for some $P \in L_0$ and $Q \in L_1$. By model (2.2), we have $U(P) + b = U(Q)$. As $\alpha P + (1-\alpha)Q \in L_1$ for all $\alpha \in (0,1)$, its evaluation should be $U(\alpha P + (1-\alpha)Q) = \alpha U(P) + (1-\alpha)U(Q)$. By $P \succ \alpha P + (1-\alpha)Q$, it must be that $U(P) + b > \alpha U(P) + (1-\alpha)U(Q)$ and hence $b > (1-\alpha)b$ for all $\alpha \in (0,1)$, which is true if and only if $b > 0$. $\square$

### 5.7 Proof of Proposition 6

*Proof.* Assume that $\succsim_1$ is more inclined to zero-risk than $\succsim_2$. Since condition (i) of Definition 3 states that $\succsim_1$ and $\succsim_2$ have the same ordering restricted to $L_1$, it follows that $u_1 \approx u_2$. Without loss of generality, set $u_1 = u_2 = u$ and assume that $b_1$ and $b_2$ refer to the constants after normalization (so that $\succsim_i$ is represented by $(u, b_i)$ for $i = 1, 2$). Since $\succsim_2$ has a nonempty preference overlapping, there exist some $P \in L_0$ and $Q \in L_1$ such that $P \sim_2 Q$. By condition (ii) of Definition 3, we must that $U(P) + b_2 = U(Q)$ and $U(P) + b_1 \geq U(Q)$, and, hence $b_1 \geq b_2$.

Conversely, assume that $u_1 \approx u_2$ and that $(u, b_i)$ represents $\succsim_i$, for $i = 1, 2$, with $b_1 \geq b_2$. Then, $\succsim_1$ and $\succsim_2$ rank lotteries in $L_1$ in the same way. Let $P \in L_0$ and $Q \in L_1$ be such that $P \succsim_2 Q$. Then, $U(P) + b_1 \geq U(P) + b_2 \geq U(Q)$, and, hence, $P \succsim_1 Q$. $\square$

### 5.8 Proof of Proposition 7

*Proof.* We shall focus on the sufficiency of the axioms, as their necessity is straightforward. Under the same arguments as in Lemma 1, axioms A1, A2 (implied by A2*), and A3 together imply the existence of vNM utility $u_0$ on $X_0$ and $u_1$ on $X$ such that $U_i$ represents $\succsim$ on $L_i$ for $i \in \{0, 1\}$. Similarly, as in Proposition 3, A3* is implied by A5 when $O_\sim$ is nontrivial, and it is otherwise directly assumed. This indicates that $u_1$ and $u_0$ can be chosen to be the same on $X_0$, which we shall do and denote it by a function $u$ defined over $X$. If the image $u(X)$ is finite, we are essentially back to the case with finite $X$. We therefore assume $u(X)$ is infinite.

We first consider the case where $O_\sim$ is empty and hence either $L_0 \succ L_1$ or $L_1 \succ L_0$. It is clear that an ascending (resp. descending) sequence of lotteries in $L_0$ exists if and only if $u(X_0)$ is unbounded from above (resp. below). And, by A3*, for any ascending (resp. descending) sequence of lotteries in $L_0$, we can mix them to a common outcome in $X \backslash X_0$ and get an ascending (resp. descending) sequence in $L_1$. So, by Archimedeanity, $u(X_0)$ has to be bounded, and, indeed, there are only three possible scenarios: $u(X)$ is bounded, or $u(X_0)$ is bounded but $u(X)$ is unbounded from above but bounded from below with $L_1 \succ L_0$, or $u(X_0)$ is bounded but $u(X)$ is unbounded from below but bounded from above with $L_0 \succ L_1$. In all three scenarios, we can find a number $b$ to separate the two sets $U(L_0)$ and $U(L_1)$ and therefore establish the model.

Suppose now that $O_\sim$ is nonempty. Let $\{x^0_{l,n}, x^0_{h,n}\}_{n=1}^\infty \subset X_0$ and $\{x^1_{l,n}, x^1_{h,n}\}_{n=1}^\infty \subset X \backslash X_0$ be such that, first, $x^i_{h,n+1} \succsim x^i_{h,n} \succsim x^i_{l,n} \succsim x^i_{l,n+1}$ for all $n \geq 1$ and $i \in \{0, 1\}$, second, $lim_{n\to\infty} u(x^0_{l,n}) = inf_{x\in X_0} u(x)$, $lim_{n\to\infty} u(x^0_{h,n}) = sup_{x\in X_0} u(x)$, $lim_{n\to\infty} u(x^1_{l,n}) = inf_{x\in X\backslash X_0} u(x)$, and $lim_{n\to\infty} u(x^1_{h,n}) = sup_{x\in X\backslash X_0} u(x)$, and, third, if $inf_{x\in X_0} u(x)$ is attainable in $X_0$, let $x^0_{l,n} \in X_0$ for any $n$ be an outcome that attains it, similarly we take $x^0_{h,n} \in X_0$ for $sup_{x\in X_0} u(x)$, $x^1_{l,n} \in X \backslash X_0$ for $inf_{x\in X\backslash X_0} u(x)$, and $x^1_{h,n} \in X \backslash X_0$ for $sup_{x\in X\backslash X_0} u(x)$, whenever possible. For all $n$, Proposition 2 is applicable to $\succsim$ restricted on $\Delta(\{x^0_{h,n}, x^0_{l,n}, x^1_{h,n}, x^1_{l,n}\})$, from which we obtain model (2.2) with some $b_n$ and the fixed $U$.

We claim that, for all $n$, representation (2.2) with $b_n$ and $U$ also works on $\Delta(X_n)$ where $X_n = \{x \in X_0 : x^0_{h,n} \succsim x \succsim x^0_{l,n}\} \cup \{x \in X \backslash X_0 : x^1_{h,n} \succsim x \succsim x^1_{l,n}\}$. This is because, for any lottery $P \in \Delta(X_n)$, we can replace any good outcome in its support by a lottery in $\Delta(\{x^0_{h,n}, x^0_{l,n}\})$ and any bad outcome by a lottery in $\Delta(\{x^1_{h,n}, x^1_{l,n}\})$, and hence we obtain an indifferent (compound) lottery in $\Delta(\{x^0_{h,n}, x^0_{l,n}, x^1_{h,n}, x^1_{l,n}\})$ that does not change $P$'s zero-risk nature and utility according to $U$.

For any $n$ such that $\succsim$ restricted on $\Delta(\{x^0_{h,n}, x^0_{l,n}, x^1_{h,n}, x^1_{l,n}\})$ has a nonempty preference overlapping, we must have $b_n = b_m$ for all $m \geq n$. This is due to the uniqueness property of parameter $b$ and the fact that the model on $\Delta(X_m)$ also works on $\Delta(X_n)$ as $X_n \subset X_m$. For the final unified model, we hence set its parameter $b$ to a $b_n$, as long as the preference overlapping on $\Delta(\{x^0_{h,n}, x^0_{l,n}, x^1_{h,n}, x^1_{l,n}\})$ is nonempty. Finally, since every lottery in $\Delta_0(X)$ is in all $\Delta(X_n)$ with large enough $n$, the unified model works on the whole $\Delta_0(X)$. $\square$

## 5.9 Proof of Theorem 2

We start with a few preliminary lemmas that will be useful to prove Theorem 2.

**Lemma 3.** *There is a linear vNM utility $U$ on $\Delta(X)$, such that, for all $\mathcal{I} \subset \mathcal{K}$, $\succsim$ restricted on $L_{\mathcal{I}}$ can be represented by $U$.*

*Proof.* By Herstein and Milnor (1953), the intra-comparison axioms imply the existence of linear utility $U_{\mathcal{I}}$ on $L_{\mathcal{I}}$ for all $\mathcal{I} \subset \mathcal{K}$. MP2 (Restricted Independence) indicates that $U_{\mathcal{K}}$ agrees with $U_{\mathcal{I}}$ on $L_{\mathcal{I}}$ for all $\mathcal{I} \subset \mathcal{K}$. Hence we simply can use $U_{\mathcal{K}}$ for the role of $U$. $\square$

For the rest of the proof, we shall often refer to the function $U$ characterized in the last lemma.

**Lemma 4.** *For all $I, I' \subset \mathcal{K}$ there exists a $b_{\mathcal{I}\mathcal{I}'} \in \mathbb{R}$ such that $U + b_{\mathcal{I}\mathcal{I}'}$ on $L_{\mathcal{I}}$ and $U$ on $L_{\mathcal{I}'}$ jointly represent $\succsim$ on $L_{\mathcal{I}} \cup L_{\mathcal{I}'}$.*

*Proof.* Following the same arguments presented in the proof of Proposition 1, MP1 (Restricted Continuity) can be shown to imply that, for all $I, I' \subset \mathcal{K}$, there exists a continuous strictly increasing function $g_{\mathcal{I}\mathcal{I}'} : U(L_{\mathcal{I}}) \to \mathbb{R}$ such that $g_{\mathcal{I}\mathcal{I}'} \circ U$ on $L_{\mathcal{I}}$ and $U$ on $L_{\mathcal{I}'}$ jointly represent $\succsim$ on $L_{\mathcal{I}} \cup L_{\mathcal{I}'}$.

If $O_{\mathcal{I}\mathcal{I}'} = \emptyset$, we can set $b_{\mathcal{I}\mathcal{I}'}$ to be any number strictly larger than $supU(L_{\mathcal{I}'}) - infU(L_{\mathcal{I}})$ if $L_{\mathcal{I}} \succ L_{\mathcal{I}'}$, and, we can take $b_{\mathcal{I}\mathcal{I}'}$ to be any number strictly smaller than $infU(L_{\mathcal{I}'}) - supU(L_{\mathcal{I}})$ if $L_{\mathcal{I}'} \succ L_{\mathcal{I}}$.

If $O_{\mathcal{I}\mathcal{I}'} \neq \emptyset$ but $P \sim Q$ for all $(P, P'), (Q, Q') \in O_{\mathcal{I}\mathcal{I}'}$, it must be that $U(P') - U(P) = U(Q') - U(Q)$. In this case, we can fix a pair $(P, P') \in O_{\mathcal{I}\mathcal{I}'}$ and set $b_{\mathcal{I}\mathcal{I}'} = U(P') - U(P)$. Notice that by construction $b_{\mathcal{I}\mathcal{I}'}$ is also equal to $supU(L_{\mathcal{I}'}) - infU(L_{\mathcal{I}})$ if $P \succ L_{\mathcal{I}'}$ for some $P \in L_{\mathcal{I}}$, and, it is equal to $infU(L_{\mathcal{I}'}) - supU(L_{\mathcal{I}})$ if $P' \succ L_{\mathcal{I}}$ for some $P' \in L_{\mathcal{I}'}$.

Finally, suppose that $O_{\mathcal{I}\mathcal{I}'} \neq \emptyset$ and $P \succ Q$ for some $(P, P'), (Q, Q') \in O_{\mathcal{I}\mathcal{I}'}$. Let $O_{\mathcal{I}} = \{P \in L_{\mathcal{I}} : (P, P') \in O_{\mathcal{I}\mathcal{I}'}\}\}$ denote the subset of $L_{\mathcal{I}}$ that has preference overlapping with $L_{\mathcal{I}'}$, and $O_{\mathcal{I}'}$ is similarly defined. We shall show that $g_{\mathcal{I}\mathcal{I}'}$ is a translation on $U(O_{\mathcal{I}})$ which implies that $g_{\mathcal{I}\mathcal{I}'}$ on its entire domain can be defined as a translation.

Let $P_u \in L_\emptyset$ denote the uniform distribution over $X_0$. For any number $a$ in the interior of $U(O_{\mathcal{I}})$ and any number $b$ in the interior of $U(O_{\mathcal{I}'})$, there exist $P \in L_{\mathcal{I}}$, $Q \in L_{\mathcal{I}'}$, and $\alpha, \beta \in (0,1)$ such that $U_{\mathcal{I}}(\alpha P_u + (1-\alpha)P) = a$ and $U_{\mathcal{I}'}(\beta P_u + (1-\beta)Q) = b$. Without loss of generality we can assume $\alpha = \beta$. For any number $a$ in the interior of $U(O_{\mathcal{I}})$, take $\alpha_a P_u + (1-\alpha_a)P \in L_{\mathcal{I}}$, $\alpha_a P_u + (1-\alpha_a)Q \in L_{\mathcal{I}'}$, and $\alpha_a \in (0,1)$ with $U(\alpha_a P_u + (1-\alpha_a)P) = a$ and $U(\alpha_a P_u + (1-\alpha_a)Q) = g_{\mathcal{I}\mathcal{I}'}(a)$. Let $H_a$ denote the open convex interval $\alpha_a (U(L_\emptyset) - U(P_u))$ excluding the boundary. By MP3 ($L_\emptyset$-Cancellation), we can conclude that

$$g_{\mathcal{I}\mathcal{I}'}(a+b) = g_{\mathcal{I}\mathcal{I}'}(a) + b, \forall b \in H_a.$$

That is, $g_{\mathcal{I}\mathcal{I}'}$ is locally a translation of $a + H_a$ to $g_{\mathcal{I}\mathcal{I}'}(a) + H_a$. Since $U(O_{\mathcal{I}})$ excluding its boundary can be expressed as $\cup_{n=1}^{\infty} C_n$, where each $C_n$ is a compact convex interval and $C_n \subset C_{n+1}$ for all $n \geq 1$. Each $C_n$ is covered by the open intervals $\{a + H_a\}_{a \in C_n}$ and hence there exists a finite sub-cover $\{a_m + H_{a_m}\}_{m=1}^{M_n}$ with $a_m < a_{m+1}$ for all $1 \leq m \leq M_n - 1$. Since in this sub-cover any $a_m + H_{a_m}$ has a nonempty intersection with some other $a_{m'} + H_{a_{m'}}$, and the intersection itself is an open interval, it must be that $g_{\mathcal{I}\mathcal{I}'}$ is a translation on $(a_m + H_{a_m}) \cup (a_{m'} + H_{a_{m'}})$. By inducting on $m$, we can conclude that $g_{\mathcal{I}\mathcal{I}'}$ is a translation on $C_n$, and hence also on $\cup_{n=1}^{\infty} C_n$. Since $g_{\mathcal{I}\mathcal{I}'}$ is continuous on $U(O_{\mathcal{I}})$, we can conclude that $g$ is a translation on $U(O_{\mathcal{I}})$. By construction we can also make $g$ a translation on the whole $U(L_{\mathcal{I}})$. □

**Lemma 5.** *For any $\mathbb{S} \subset 2^{\mathcal{K}}$, if $\{L_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}}$ is connected, then there exist a set of real numbers $\{b_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}}$ such that $\{U + b_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}}$ jointly represent $\succsim$ on $\cup_{\mathcal{I} \in \mathbb{S}} L_{\mathcal{I}}$ . Moreover, given $U$, $\{b_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}}$ is unique up to (constant) translations.*

*Proof.* We shall call any $L_{\mathcal{I}}$ a *segment* and a set of segments a *stream*. For any segment $L_{\mathcal{I}}$, *the right stream* of $L_{\mathcal{I}}$ refers to the set $\{L_{\mathcal{I}'} \in \{L_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}} : \exists P' \in L_{\mathcal{I}'}, P' \succ L_{\mathcal{I}}; \nexists Q' \in L_{\mathcal{I}'}, L_{\mathcal{I}} \succ Q'\}$ and *the left stream* of $L_{\mathcal{I}}$ refers to the set $\{L_{\mathcal{I}'} \in \{L_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}} : \exists P' \in L_{\mathcal{I}'}, L_{\mathcal{I}} \succ P'; \nexists Q' \in L_{\mathcal{I}'}, Q' \succ L_{\mathcal{I}}\}$.

Fix an $\mathcal{I}_0 \in \mathbb{S}$ such that there is no $L_{\mathcal{I}}$ with $P, Q \in L_{\mathcal{I}}$ such that $P \succ L_{\mathcal{I}_0} \succ Q$. Let $\mathcal{O}^{(0)} = \{L_{\mathcal{I}_0}\}$ and, for all $n \geq 1$, let $\mathcal{O}^{(n)} \subset \{L_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}}$ denote the collection of segments connected to some segment in $\mathcal{O}^{(n-1)}$ that are not in $\cup_{k=0}^{n-1} \mathcal{O}^{(k)}$. Every $L_{\mathcal{I}} \in \{L_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}}$ is in $\mathcal{O}^{(n)}$ for some $n$. The reason is that, by the connectedness of $\{L_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}}$, there is a connected sequence $\{L_{\mathcal{I}(n)}\}_{n=1}^{N}$ starting from $L_{\mathcal{I}_0}$ and ending with $L_{\mathcal{I}}$. If $N = 1$, $L_{\mathcal{I}} = L_{\mathcal{I}_0} \in \mathcal{O}^{(0)}$, and, if $N = 2$, $L_{\mathcal{I}} \in \mathcal{O}^{(1)}$. For length $N = n$, we can suppose that $L_{\mathcal{I}(n-1)} \in \mathcal{O}^{(m)}$ for some $m \leq n-2$. By construction any segment that is connected to some segment in $\mathcal{O}^{(m)}$ should be in $\mathcal{O}^{(m-1)} \cup \mathcal{O}^{(m)} \cup \mathcal{O}^{(m+1)}$, hence it must be that $L_{\mathcal{I}(n)} \in \mathcal{O}^{(m-1)} \cup \mathcal{O}^{(m)} \cup \mathcal{O}^{(m+1)}$.

We now state, and shortly prove some structural properties of the $\mathcal{O}^{(n)}$'s.

*Property O1*: Consider any connected sequence $\{L_{\mathcal{I}(n)}\}_{n=1}^{N}$ with $L_{\mathcal{I}(n)} \in \mathcal{O}^{(n-1)}$, $1 \leq$

$n \leq N$. If $L_{\mathcal{I}(2)}$ is in the right stream of $L_{\mathcal{I}(1)}$, then $L_{\mathcal{I}(n)}$ is in the right stream of $L_{\mathcal{I}(n-1)}$ for all $3 \leq n \leq N-1$, while, if $L_{\mathcal{I}(2)}$ is in the left stream of $L_{\mathcal{I}(1)}$, $L_{\mathcal{I}(n)}$ is in the left stream of $L_{\mathcal{I}(n-1)}$ for all $3 \leq n \leq N-1$.

*Property O2.1*: For every $\mathcal{O}^{(n)}$ with $n \geq 0$, there exists an $L_{l,n} \in \mathcal{O}^{(n)}$, which we shall call *the left reach* of $\mathcal{O}^{(n)}$, and an $L_{r,n} \in \mathcal{O}^{(n)}$, *the right reach* of $\mathcal{O}^{(n)}$, such that every $L_{\mathcal{I}} \in \mathcal{O}^{(n+1)}$ in the left stream of $L_{\mathcal{I}_0}$ is connected to $L_{l,n}$ and every $L_{\mathcal{I}} \in \mathcal{O}^{(n+1)}$ in the right stream of $L_{\mathcal{I}_0}$ is connected to $L_{r,n}$.

*Property O2.2*: For every $\mathcal{O}^{(n)}$ with $n \geq 0$, and any two $L_{\mathcal{I}}, L_{\mathcal{I}'} \in \mathcal{O}^{(n)}$ that are connected to some segments in $\mathcal{O}^{(n+1)}$, if both $L_{\mathcal{I}}$ and $L_{\mathcal{I}'}$ are in the right stream of $L_{\mathcal{I}_0}$ or in the left stream of $L_{\mathcal{I}_0}$, then they are connected and one of them is connected to a (weakly) larger set of segments in $\mathcal{O}^{(n+1)}$.

*Proof of O1*: We assume $N \geq 4$ for the claim to have any bite. Notice that $L_{\mathcal{I}(1)} = L_{\mathcal{I}_0}$ by construction and, as long as $N \geq 3$, $L_{\mathcal{I}(2)}$ has to be in either the left or the right stream of $L_{\mathcal{I}(1)}$. We prove by induction starting with $n = 3$ and suppose $L_{\mathcal{I}(2)}$ is in the right stream of $L_{\mathcal{I}(1)}$. If $U(L_{\mathcal{I}(3)}) + b_{\mathcal{I}(3)\mathcal{I}(2)}$ is contained in $U(L_{\mathcal{I}(2)})$, every segment that overlaps with $L_{\mathcal{I}(3)}$ would also overlap with $L_{\mathcal{I}(2)}$, which implies that $L_{\mathcal{I}(4)}$ should be in $\mathcal{O}^{(2)}$ as opposed to $\mathcal{O}^{(3)}$, a contradiction. If $U(L_{\mathcal{I}(3)}) + b_{\mathcal{I}(3)\mathcal{I}(2)}$ contains $U(L_{\mathcal{I}(2)})$ or $L_{\mathcal{I}(3)}$ is in the left stream of $L_{\mathcal{I}(2)}$, then $L_{\mathcal{I}(3)}$ would overlap with $L_{\mathcal{I}(1)}$ and hence it would be in $\mathcal{O}^{(1)}$, contradicting $L_{\mathcal{I}(3)} \in \mathcal{O}^{(2)}$. So, $L_{\mathcal{I}(3)}$ can only be in the right stream of $L_{\mathcal{I}(2)}$. Similar arguments apply for the $n$th step, given $L_{\mathcal{I}(n-1)}$ is in the right stream of $L_{\mathcal{I}(n-2)}$. The case of $L_{\mathcal{I}(2)}$ being in the left stream of $L_{\mathcal{I}(1)}$ is similar. Notice that the claim excludes $n = N$ because $U(L_{\mathcal{I}(N)}) + b_{\mathcal{I}(N)\mathcal{I}(N-1)}$ could be contained in $U(L_{\mathcal{I}(N-1)})$ (and hence $L_{\mathcal{I}(N)}$ is not connected to any segment in $\mathcal{O}^{(N)}$).

*Proof of O2.1 and O2.2*: The two properties shall be proved by induction, using O2.1 of step $n-1$ to prove O2.2 of step $n$, which is then further used to prove O2.1 of step $n$. For $n = 0$, $\mathcal{O}^{(0)}$ is a singleton and the two properties hold trivially with $L_{l,0} = L_{r,0} = L_{\mathcal{I}_0}$. Assume the two properties for step $n-1$ and consider any two $L_{\mathcal{I}}, L_{\mathcal{I}'} \in \mathcal{O}^{(n)}$ that are connected to some segments in $\mathcal{O}^{(n+1)}$. If they are in the right stream of $L_{\mathcal{I}_0}$, then by property O2.1 of step $n-1$ they are connected to $L_{r,n-1}$ and by property O1 they are also in the right stream of $L_{r,n-1}$. Hence they have to be connected and the one that reaches further to the right (i.e., it is $L_{\mathcal{I}}$ if there is a $P \in L_{\mathcal{I}}$ with $P \succ L_{\mathcal{I}'}$, and, both shall work if they have the same reach to the right) is connected to a larger set of segments in $\mathcal{O}^{(n+1)}$. The right reach $L_{r,n}$ can be found out by repeatedly using O2.2 of step $n$ to compare pairs of segments in $\mathcal{O}^{(n)}$ that are connected to some segments in $\mathcal{O}^{(n+1)}$ and are also in the right stream of $L_{\mathcal{I}_0}$, picking the one that reaches further to the right in each round of comparison. The left reach $L_{l,n}$ can be identified analogously.

We now establish a 3-segment property and a 4-segment property that are useful in identifying $\{b_{\mathcal{I}}\}_{\mathcal{I} \in \mathbb{S}}$. The 3-segment property is that, if three segments $L_{\mathcal{I}}, L_{\mathcal{J}}, L_{\mathcal{H}}$ are pairwise connected, then $b_{\mathcal{I}\mathcal{H}} = b_{\mathcal{I}\mathcal{J}} + b_{\mathcal{J}\mathcal{H}}$. Pairwise connectedness implies that there must exist $P \in L_{\mathcal{I}}, Q \in L_{\mathcal{J}}, R \in L_{\mathcal{H}}$ with $P \sim Q \sim R$, which indicates that $U(P) + b_{\mathcal{I}\mathcal{H}} = U(R)$, $U(P) + b_{\mathcal{I}\mathcal{J}} = U(Q)$, $U(Q) + b_{\mathcal{J}\mathcal{H}} = U(R)$, and hence $b_{\mathcal{I}\mathcal{H}} = b_{\mathcal{I}\mathcal{J}} + b_{\mathcal{J}\mathcal{H}}$. The 4-segment property is that, if $L_{\mathcal{I}_1}, L_{\mathcal{I}_2}, L_{\mathcal{I}'_2}$ are pairwise connected and $L_{\mathcal{I}_2}, L_{\mathcal{I}'_2}, L_{\mathcal{I}_3}$ are pairwise connected, then $b_{\mathcal{I}_1\mathcal{I}_2} + b_{\mathcal{I}_2\mathcal{I}_3} = b_{\mathcal{I}_1\mathcal{I}'_2} + b_{\mathcal{I}'_2\mathcal{I}_3}$. The 3-segment property implies that $b_{\mathcal{I}_1\mathcal{I}'_2} + b_{\mathcal{I}'_2\mathcal{I}_2} = b_{\mathcal{I}_1\mathcal{I}_2}$ and $b_{\mathcal{I}'_2\mathcal{I}_3} = b_{\mathcal{I}'_2\mathcal{I}_2} + b_{\mathcal{I}_2\mathcal{I}_3}$, which by adding up the two equations imply $b_{\mathcal{I}_1\mathcal{I}_2} + b_{\mathcal{I}_2\mathcal{I}_3} = b_{\mathcal{I}_1\mathcal{I}'_2} + b_{\mathcal{I}'_2\mathcal{I}_3}$.

The set $\{b_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}}$ is defined as follows. Let $b_{\mathcal{I}_0} = 0$. For any $L_{\mathcal{I}} \in O^{(N)}$, let $b_{\mathcal{I}} = \sum_{n=1}^{N} b_{\mathcal{I}(n+1)\mathcal{I}(n)}$, where $\{L_{\mathcal{I}(n)}\}_{n=1}^{N+1}$ is a connected sequence with $L_{\mathcal{I}(1)} = L_{\mathcal{I}_0}$, $L_{\mathcal{I}(N+1)} = L_{\mathcal{I}}$, and $L_{\mathcal{I}(n)} \in \mathcal{O}^{(n-1)}$, $1 \leq n \leq N+1$. For it to be a valid definition, we claim that if two such connected sequences $\{L_{\mathcal{I}(n)}\}_{n=1}^{N+1}$ and $\{L_{\mathcal{I}'(n)}\}_{n=1}^{N+1}$ exist, it must be that $\sum_{n=1}^{N} b_{\mathcal{I}(n+1)\mathcal{I}(n)} = \sum_{n=1}^{N} b_{\mathcal{I}'(n+1)\mathcal{I}'(n)}$. Since $L_{\mathcal{I}(N+1)} = L_{\mathcal{I}'(N+1)} = L_{\mathcal{I}}$ and by property O2.2, $L_{\mathcal{I}(N)}$ and $L_{\mathcal{I}'(N)}$ must be connected and they are both connected to one of $L_{\mathcal{I}(N-1)}$ and $L_{\mathcal{I}'(N-1)}$, which we without loss of generality take to be $L_{\mathcal{I}(N-1)}$. The 4-segment property leads to $b_{\mathcal{I}'(N+1)\mathcal{I}'(N)} + b_{\mathcal{I}'(N)\mathcal{I}(N-1)} = b_{\mathcal{I}(N+1)\mathcal{I}(N)} + b_{\mathcal{I}(N)\mathcal{I}(N-1)}$ and hence we only need to show $b_{\mathcal{I}'(N)\mathcal{I}(N-1)} + \sum_{n=1}^{N-2} b_{\mathcal{I}(n+1)\mathcal{I}(n)} = \sum_{n=1}^{N-1} b_{\mathcal{I}'(n+1)\mathcal{I}'(n)}$, that is, the two sequences can be thought to be one segment shorter, as the original final segment $L_{\mathcal{I}(N+1)} = L_{\mathcal{I}'(N+1)} = L_{\mathcal{I}}$ is now replaced by $L_{\mathcal{I}'(N)}$. We can repeat this process till $N = 2$, for which using the 4-segment property once more delivers the claim. Set $\{b_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}}$ is unique under $b_{\mathcal{I}_0} = 0$ because $b_{\mathcal{I}\mathcal{J}}$ is unique given $U$ for all $\mathcal{I}, \mathcal{J} \in \mathbb{S}$, and, in general, the set is unique up to a constant translation given $U$.

We claim that $\{U + b_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}}$ jointly represent $\succsim$ on $\cup_{\mathcal{I}\in\mathbb{S}} L_{\mathcal{I}}$. For all $L_{\mathcal{I}}, L_{\mathcal{J}} \in \{L_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}}$ that are connected, they are in either the same $\mathcal{O}^{(n)}$ or some consecutive $\mathcal{O}^{(n)}$ and $\mathcal{O}^{(n+1)}$, and, we need to show $b_{\mathcal{I}} - b_{\mathcal{J}} = b_{\mathcal{I}\mathcal{J}}$. If $L_{\mathcal{I}} \in \mathcal{O}^{(n)}$ and $L_{\mathcal{J}} \in \mathcal{O}^{(n+1)}$ for some $n$, property O2.1 implies the existence of a sequence $\{L_{\mathcal{I}(n)}\}_{n=1}^{N+1}$ with $L_{\mathcal{I}(1)} = L_{\mathcal{I}_0}$, $L_{\mathcal{I}(N)} = L_{\mathcal{I}}$, $L_{\mathcal{I}(N+1)} = L_{\mathcal{J}}$, and $L_{\mathcal{I}(n)} \in \mathcal{O}^{(n-1)}$ for all $1 \leq n \leq N+1$, which directly implies $b_{\mathcal{J}} = b_{\mathcal{I}} + b_{\mathcal{J}\mathcal{I}}$ (and hence $b_{\mathcal{I}} - b_{\mathcal{J}} = b_{\mathcal{I}\mathcal{J}}$). If $L_{\mathcal{I}}, L_{\mathcal{J}} \in \mathcal{O}^{(n)}$, then property O2.1 implies the existence of a $L_{\mathcal{H}} \in \mathcal{O}^{(n-1)}$ that is connected to both $L_{\mathcal{I}}$ and $L_{\mathcal{J}}$. Because property O2.1 also implies the existence of a sequence $\{L_{\mathcal{I}(n)}\}_{n=1}^{N+1}$ with $L_{\mathcal{I}(1)} = L_{\mathcal{I}_0}$, $L_{\mathcal{I}(N)} = L_{\mathcal{H}}$, $L_{\mathcal{I}(N+1)} = L_{\mathcal{I}}$, and $L_{\mathcal{I}(n)} \in \mathcal{O}^{(n-1)}$ for all $1 \leq n \leq N+1$, it must be that $b_{\mathcal{I}} = b_{\mathcal{H}} + b_{\mathcal{I}\mathcal{H}}$. Similarly, it must be $b_{\mathcal{J}} = b_{\mathcal{H}} + b_{\mathcal{J}\mathcal{H}}$. By taking difference of the two equations we get $b_{\mathcal{I}} - b_{\mathcal{J}} = b_{\mathcal{I}\mathcal{H}} - b_{\mathcal{J}\mathcal{H}}$, which by the 3-segment property implies $b_{\mathcal{I}} - b_{\mathcal{J}} = b_{\mathcal{I}\mathcal{J}}$. For all $L_{\mathcal{I}}, L_{\mathcal{J}} \in \{L_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}}$ that are not connected, there is a sequence starting with $L_{\mathcal{I}}$, ending with $L_{\mathcal{J}}$, and any two segments in the sequence are connected if and only if they are consecutive. The linear structure and the result on connected segments imply that $U + b_{\mathcal{I}}$ on $L_{\mathcal{I}}$ and $U + b_{\mathcal{J}}$ on $L_{\mathcal{J}}$ jointly represent $\succsim$ on $L_{\mathcal{I}} \cup L_{\mathcal{J}}$. ◻

We can now establish Theorem 2.

*Proof.* The whole collection $\{L_{\mathcal{I}}\}_{\mathcal{I}\subset K}$ can be partitioned into $\{\{L_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}_n}\}_{n=1}^{N}$, where, for each $n$, $\{L_{\mathcal{I}}\}_{\mathcal{I}\in\mathbb{S}_n}$ is a collection of connected segments and, for all $n \neq m$, every segment in

$\{L_\mathcal{I}\}_{\mathcal{I}\in\mathbb{S}_n}$ is not connected to any segment in $\{L_\mathcal{J}\}_{\mathcal{J}\in\mathbb{S}_m}$. For all $n \neq m$, $L_\mathcal{I} \in \{L_\mathcal{I}\}_{\mathcal{I}\in\mathbb{S}_n}$, and $L_\mathcal{J} \in \{L_\mathcal{J}\}_{\mathcal{J}\in\mathbb{S}_m}$, $L_\mathcal{I} \succ L_\mathcal{J}$ implies $L_\mathcal{I} \succ L_{\mathcal{J}'}$ for all $L_{\mathcal{J}'} \in \{L_\mathcal{J}\}_{\mathcal{J}\in\mathbb{S}_m}$ that is connected to $L_\mathcal{J}$. Using this property throughout $\{L_\mathcal{J}\}_{\mathcal{J}\in\mathbb{S}_m}$, we can conclude that $L_\mathcal{I} \succ \{L_\mathcal{J}\}_{\mathcal{J}\in\mathbb{S}_m}$, and then using it throughout $\{L_\mathcal{I}\}_{\mathcal{I}\in\mathbb{S}_n}$ we get $\{L_\mathcal{I}\}_{\mathcal{I}\in\mathbb{S}_n} \succ \{L_\mathcal{J}\}_{\mathcal{J}\in\mathbb{S}_m}$. Therefore, we can without loss of generality assume that $\{L_\mathcal{I}\}_{\mathcal{I}\in\mathbb{S}_{n+1}} \succ \{L_\mathcal{I}\}_{\mathcal{I}\in\mathbb{S}_n}$ for all $1 \leq n \leq N-1$. For each $\mathbb{S}_n$, we obtain $\{U + b'_\mathcal{I}\}_{\mathcal{I}\in\mathbb{S}_n}$ that jointly represent $\succsim$ on $\cup_{\mathcal{I}\in\mathbb{S}_n} L_\mathcal{I}$. Let $g_n$ be any number strictly larger than $sup \cup_{\mathcal{I}\in\mathbb{S}_n} (U(L_\mathcal{I}) + b'_\mathcal{I}) - inf \cup_{\mathcal{I}\in\mathbb{S}_{n+1}} (U(L_\mathcal{I}) + b'_\mathcal{I})$, which is the minimum utility gap between $\mathbb{S}_n$ and $\mathbb{S}_{n+1}$, $1 \leq n \leq N-1$. For the unified representation, let $b_\mathcal{I} = b'_\mathcal{I} + \sum_{m=1}^{n-1} g_m$ if $\mathcal{I} \in \mathbb{S}_n$. □

### 5.10 Proof of Proposition 8

*Proof.* The necessity of the additivity condition is obvious. For sufficiency, we shall show that $b_{\mathcal{I}\cup\mathcal{I}'} = b_\mathcal{I} + b_{\mathcal{I}'}$ for all disjoint $\mathcal{I}, \mathcal{I}' \subset \mathcal{K}$. By connectedness there exists a sequence $\{L_{\mathcal{I}(n)}\}_{n=1}^{N_1}$ with $L_{\mathcal{I}(1)} = L_\mathcal{I}$, $L_{\mathcal{I}(N_1)} = L_{\mathcal{I}\cup\mathcal{I}'}$, and $L_{\mathcal{I}(n)}$ is connected to $L_{\mathcal{I}(n+1)}$, $1 \leq n \leq N_1 - 1$. As $L_{\mathcal{I}(n)}$ is connected to $L_{\mathcal{I}(n+1)}$, there exist $P_n \in L_{\mathcal{I}(n)}$ and $Q_n \in L_{\mathcal{I}(n+1)}$ with $P_n \sim Q_n$, for all $1 \leq n \leq N_1 - 1$. Similarly, there exists a sequence $\{L_{\mathcal{I}'(n)}\}_{n=1}^{N_2}$ that connects $L_{\mathcal{I}'}$ to $L_{\mathcal{I}_\emptyset}$, with $P'_n \in L_{\mathcal{I}'(n)}$ and $Q'_n \in L_{\mathcal{I}'(n+1)}$ such that $P'_n \sim Q'_n$, for all $1 \leq n \leq N_2 - 1$.

Consider $(P_n, Q_n)_{n=1}^{N_1-1}$ and $(P'_n, Q'_n)_{n=1}^{N_2-1}$ as a whole finite collection of indifferences. Notice that every $\mathcal{I}(n)$ or $\mathcal{I}'(n)$ appears twice in the collection, once on the left "$P$" lottery side and once on the right "$Q$" lottery side, except that, $\mathcal{I}$ appears only once, as $P_1 \in L_\mathcal{I}$ on the left, $\mathcal{I}'$ appears once, as $P'_1 \in L_{\mathcal{I}'}$ on the left, $\mathcal{I}\cup\mathcal{I}'$ appears once, as $Q_{N_1-1} \in L_{\mathcal{I}\cup\mathcal{I}'}$ on the right, and, $\emptyset$ appears once, as $Q'_{N_2-1} \in L_\emptyset$ on the right. Formally, we have

$$\sum_{n=1}^{N_1-1} 1_{\mathcal{I}(n)} + \sum_{n=1}^{N_2-1} 1_{\mathcal{I}'(n)} = \sum_{n=2}^{N_1} 1_{\mathcal{I}(n)} + \sum_{n=2}^{N_2} 1_{\mathcal{I}'(n)}. \tag{5.1}$$

Additivity is hence applicable and it delivers the indifference between lottery

$$\sum_{n=1}^{N_1-1} \frac{1}{N_1+N_2-2} P_n + \sum_{n=1}^{N_2-1} \frac{1}{N_1+N_2-2} P'_n$$

and lottery

$$\sum_{n=1}^{N_1-1} \frac{1}{N_1+N_2-2} Q_n + \sum_{n=1}^{N_2-1} \frac{1}{N_1+N_2-2} Q'_n.$$

The two lotteries violate the same set of principles $\left(\cup_{n=1}^{N_1} \mathcal{I}(n)\right) \cup \left(\cup_{n=1}^{N_2} \mathcal{I}'(n)\right)$, implied by Equation 5.1. Hence the indifference translates into

$$\sum_{n=1}^{N_1-1} U(P_n) + \sum_{n=1}^{N_2-1} U(P'_n) = \sum_{n=1}^{N_1-1} U(Q_n) + \sum_{n=1}^{N_2-1} U(Q'_n)$$

in utility terms. This, along with the collection of indifferences, imply

$$\sum_{n=1}^{N_1-1} b_{\mathcal{I}_n} + \sum_{n=1}^{N_2-1} b_{\mathcal{I}'_n} = \sum_{n=2}^{N_1} b_{\mathcal{I}_n} + \sum_{n=2}^{N_2} b_{\mathcal{I}'_n},$$

which, after cancellation of common terms, yields $b_{\mathcal{I}\cup\mathcal{I}'} = b_{\mathcal{I}} + b_{\mathcal{I}'}$. □