Memorable consumption ☆

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Abstract

People often consume non-durable goods in a way that seems inconsistent with preferences for smoothing consumption over time. We suggest that such patterns of consumption can be better explained if one takes into account the future utility flows generated by memorable consumption goods—goods, such as a honeymoon or a vacation, whose utility flow outlives their physical consumption. We consider a model in which a consumer enjoys current consumption as well as utility generated by earlier memorable consumption. Lasting utility flows are generated only by some goods, and only when their consumption exceeds customary levels by a sufficient margin. We offer axiomatic foundations for the structure of the utility function and study optimal consumption in a dynamic model. We show that rational consumers, taking into account future utility flows, would make optimal choices that rationalize lumpy patterns of consumption.
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1. Introduction

1.1. Consumption patterns

The conceptual point of departure for modeling intertemporal consumption is a utility function of the form

$$U(c_0, c_1, \ldots, c_T) = \sum_{t=0}^{T} \delta^t u(c_t), \tag{1}$$

where c_t is consumption in period t, δ is the (stationary) discount factor, u is the (stationary) utility function, typically assumed to be concave, and T may be finite or infinity. The discounting built into this model creates an incentive to move consumption toward the present, while the ability to earn a return on unspent income creates a countervailing force. These are typically balanced by perfectly smoothing consumption.

In contrast, when asked whether they would prefer an increasing intertemporal consumption stream, such as (10, 12, 14), or the analogous decreasing consumption stream (14, 12, 10), people often prefer the first. Indeed, Kahneman and Tversky (1979) have emphasized that people often react to changes in consumption more than to absolute levels. In line with previous contributions (Helson, 1947; Markowitz, 1952), they suggest that people form reference points and evaluate current consumption relative to these reference points. This idea is consistent with modifications of the standard model according to which the consumer is viewed as maximizing

$$U(c_0, c_1, \ldots, c_T) = \sum_{t=0}^{T} \delta^t u(c_t, \Lambda_t), \tag{2}$$

where Λ_t designates a reference point that is determined (at least partially) by past consumption levels $(c_0, c_1, \ldots, c_{t-1})$.[1]

Elaborations of the standard model along the lines of (2) encounter difficulties when confronted by an example of a young couple who (not atypically) spend a quarter of their combined annual income on a wedding, as well as similar examples involving vacations, celebrations, and other seemingly nondurable consumption goods. Such a large expenditure at the very beginning of their life as a couple seems to violate the preference for consumption smoothing generated by (1). It also runs contrary to the optimal management of one's reference point that arises out of (2) and a preference for an increasing consumption stream: the more spectacular the honeymoon, the bleaker will future consumption appear in comparison. The literature is less forthcoming with a model in this case. Our view is that (1) and (2) both fail to capture the effect of memorable consumption goods. When a couple gets married, they can already envisage themselves leafing through their wedding albums in the near future, telling their children about their honeymoon in the more distant future, and generally deriving pleasure from their consumption long after it has physically ended. Indeed, the unusually large wedding expenditure is an essential ingredient in generating the utility that the couple will enjoy later—it is important that the festivities lie suffi-

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[1] Many reference-point models of consumption assume in addition that people are loss averse, in the sense that losses are felt more keenly than gains, i.e., the left derivative of u in its first argument is larger than the right derivative, when both are evaluated at $c_t = \Lambda_t$. Loss aversion does not play a role in our analysis, and we assume the function u is differentiable.

ciently outside their ordinary experience—and a substantial part of the cost is typically devoted to items (such as photography and keepsakes) designed to reinforce such future utility flows.

We analyze a model of dynamic consumer choice centered around these two key links between past consumption and subsequent utility, namely the effect of past consumption in generating future utility flows and the effect of past consumption in determining reference or "customary" consumption levels that help set the bar for generating more such flows. We seek a minimal extension of the standard dynamic choice model given by (1). This makes it straightforward to identify and quantify the differences in consumer behavior that arise because past consumption affects future utility, and to link these differences to the features of the model. Our model also retains the tractability of the standard model, positioning the model for use in applied work, as in Hai et al. (2014).

We make specific assumptions about the particular way that past consumption affects future utilities. These assumptions lie behind the tractability of our model, and it is important to understand how restrictive they are. We accordingly provide an axiomatization of the assumed form of preferences over ordinary and memorable goods that provides a foundation on which we can impose functional form assumptions.

1.2. Modeling consumption

As with any model of decision making, we face a standard interpretational question: is this a revealed-preference exercise, designed only to provide a characterization of behavior? Or is the model intended to capture also some of the mental processes underlying the choices it describes?

Our ultimate goal is to describe economic behavior. A mathematical model that could provide perfect quantitative predictions of consumer choice based on observable data would serve this purpose, whether it had a claim also to describe mental processes or not. However, such models do not yet exist. Whether one attempts to develop them, or whether one is willing to make do with models that provide qualitative insights, we believe that looking at mental processes could be a source of useful insights.

We accept it as a scientific fact that consumption generates chemical flows in the brain that provide the motivation for peoples' decisions. Moreover, we believe it to be obvious that there is a temporal dimension to these flows.[2] One enjoys a meal not simply during the instant it is consumed but for some time afterward. One can end a day content in the glow of the utility generated by activities undertaken during the day. This temporal dimension is typically suppressed in models of consumption, while our point of departure is the belief that memorable goods can generate flows of utility that are sufficiently long-lasting as to have a significant effect on consumption patterns.

We emphasize, however, that our goal is to model and study the consumption of memorable goods, not to examine the chemistry of the brain or even produce a high-level model of memory. Why do certain, sometimes seemingly trivial episodes from our childhood remain vivid memories? Why is it so easy to remember some songs, while our memories of even important events in our lives are often imprecise? These are fascinating questions, but we do not address them. Rather, our aim is to examine the economic implications of the assumption that the consumption of some goods generates long-lasting utility flows.

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[2] See, for example, Speer et al. (2014), who examine the brain activity associated with remembered events. Zauberman et al. (2009) explore some of the implications of utility flows associated with past consumption.

1.3. Relation to the literature

The first component of our model, in the form of a suggestion that one can get pleasure in the future from consumption in the past, appears early in economics, dating back (at least) to Adam Smith's (2009, p. 152) observation that "We can entertain ourselves with memories of past pleasures..." Strotz (1955) was one of the first to incorporate the utility from past consumption in a model of utility maximization, though as the title of his classic paper on dynamic consistency suggests, his model gives rise to a time inconsistency problem that renders the very notion of utility maximization ambiguous. No such difficulties arise in our setting.

The second pillar of our model is the idea that a consumer develops a notion of customary consumption that affects her current well-being and that depends on past consumption. This idea is widespread, and is perhaps most familiar from models of habit formation (see Attanasio, 1999 for a survey). As we explain in Section 5.1, the most general form of our model contains habit-formation and similar models as a special case. The primary difference in implications is that habit-formation and related models typically reinforce motives for consumption smoothing, whereas memorable goods give rise to occasional spikes in consumption.

This paper is closely related to Hai et al. (2014), who introduce the notion of memorable goods and examine the implications of memorable goods for evaluating the (excess) volatility of consumption. Hai, Krueger and Postlewaite set out a model in which past consumption affects the future through the two channels in our model. Their empirical results provide support for the importance of memorable goods. In particular, they argue that the excess sensitivity to foreseen income shocks identified by Souleles (1999) is largely due to expenditures on memorable goods. As we explain in Section 2.1 below, Hai, Krueger and Postlewaite impose a stronger separability condition on the utility implications of the consumption of ordinary and memorable goods than we do. Section 3 explains how our more general model captures an interaction between the consumption of ordinary and memorable goods that consequently does not appear in Hai, Krueger and Postlewaite. At the same time, as we explain in Section 2.2, Hai, Krueger and Postlewaite impose fewer assumptions on the relationship between the utility implications of the current and past consumption of memorable goods. Our stronger assumptions bring rewards in terms of tractability, while the basic nature of our results continue to hold in the model examined by Hai, Krueger and Postlewaite.

Hindy et al. (1992) examine a continuous-time model with a single good in which utility at time t is a function of the weighted average of consumption up to time t, including as a special case the familiar separable formulation in which utility at time t depends only on consumption at time t. Hindy et al. (1992, Section 8.3) present an example that admits a simple closed-form solution. Ingersoll (1992) and Bank and Riedel (2000) characterize the optimal consumer behavior for such models in the absence of uncertainty, while Bank and Riedel (2001) extend the analysis to uncertainty. These models contain no counterpart of our assumption that some (namely memorable) consumption goods generate continuing flows of utility if and only if their consumption sufficiently exceeds some customary level.

As Bank and Riedel (2001, p. 751) explain, as time periods grow short, and especially in the continuous-time limit, it is natural to model current utility as depending on past consumption. Suppose, for example, that one is interested in studying the consumption of food, and that food can be consumed in the form of meals that take one hour each. Suppose we work with a separable model in which utility is given by $\sum_{t=0}^{\infty} \delta^t u(c_t)$, where the length of a time period is an hour and c_t is the quantity of food consumed in period t, u is increasing, and δ is both the discount factor and the interest factor in the consumer's budget constraint. Then if u is strictly concave, the

consumer will prefer to smooth consumption, eating the same amount of food each hour. If u is linear, the consumer will be indifferent over when she consumes, and would just as well start at midnight on each Saturday and eat twenty-one meals in the subsequent twenty-one hours as have three meals a day throughout the week. If u is convex, the consumer will constantly endeavor to put off meals. Each of these outcomes has its own problematic features.

The obvious response is that we have chosen the time period incorrectly, and in particular have chosen a period length too short for separability to be reasonable. One's utility from eating typically depends on whether one has eaten in the previous hour. One then might hope to work with a separable model of utility with discrete periods *if* the period length is appropriately chosen. In the case of food, it is often (but not always, as this would not be a good model of famine) a reasonable approximation that the utility of consumption today is unaffected by consumption yesterday, and so one can take the period to be a day. However, the utility of a day's vacation typically depends on what one has been doing in recent weeks or months rather than simply the previous day, and a longer time period would be required to make separability a reasonable assumption. In general, one can always make separability a better approximation for more goods by working with longer periods. Unfortunately, as the length of a period grows, questions about the allocation of consumption within the period become more important, but lie beyond the reach of the model. In the extreme, one could take the length of a period to be a lifetime, ensuring that separability is trivial, but precluding the analysis of many interesting intertemporal patterns. Our view is that it is most useful to work with periods long enough that the utility for many goods (such as food) is reasonably modeled as separable, but short enough that nonseparabilities arise in other goods that we capture by modeling those goods as memorable goods.

Gilovich et al. (2015) argue that people derive greater satisfaction from experiential purchases than from material purchases. Experiential purchases are those that provide "lasting contributions to well being," and hence share some of the features of our memorable goods. However, their arguments are quite different, resting on the fact that experiential goods enhance social relations, contribute to one's identity, and evoke fewer social comparisons than material goods, rather than any appeal to future utility flows.

2. The model

The following four subsections develop our model. Section 2.1 introduces a distinction between two types of goods that we will refer to as ordinary goods and memorable goods. Section 2.2 introduces the structure that motivates the characterization of the latter as memorable goods. Section 2.3 shows that the resulting utility maximization problem has a solution. Section 2.4 rearranges the utility representation into a more useful form.

2.1. Ordinary goods and memorable goods

The point of departure for our model is a distinction between two types of goods, which we refer to as an ordinary good (good 1) and a memorable good (good 2). We consider a consumer who consumes these two goods in each of periods $t = 0, 1, 2, \ldots$, and we refer to $x_{it} \in \mathbb{R}_+$ ($i = 1, 2$) as the quantity of good i consumed in period t. The utility in period t depends on current consumption of good 1 but also on all past consumption of good 2. That is, utility in period t is given by a function

$$\tilde{u}_t(x_{1t}, x_{20}, \ldots, x_{2t}). \tag{3}$$

We will assume (and derive axiomatically in Section 4) a decomposition of the function $\tilde{u}_t$ as

$$\tilde{u}_t(x_{1t}, x_{20}, \ldots, x_{2t}) = u(x_{1t}, x_{2t}) + \tilde{v}_t(x_{20}, \ldots, x_{2t}), \tag{4}$$

where $u : \mathbb{R}_+^2 \to \mathbb{R}$ and $\tilde{v}_t : \mathbb{R}_+^{t+1} \to \mathbb{R}$. We could generalize the analysis to the case in which good 1 is a bundle of ordinary goods and good 2 is a bundle of memorable goods, with the notation and arguments growing more tedious in the process.

The intertemporal objective is the discounted sum of the functions $\tilde{u}(x_{1t}, x_{20}, \ldots, x_{2t})$ given by (4). We focus on the case of an infinite horizon, while noting that the adaptation to a finite horizon is straightforward (as illustrated in Section 3.3). Let $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ denote a pair of infinite sequences $(x_{10}, x_{11}, \ldots, x_{20}, x_{21}, \ldots)$ of quantities of the two consumption goods. Then the intertemporal objective is given by $U(\boldsymbol{x})$, where

$$U(\boldsymbol{x}) = \sum_{t=0}^{\infty} \delta^t \left[u(x_{1t}, x_{2t}) + \tilde{v}_t(x_{20}, \ldots, x_{2t}) \right]. \tag{5}$$

Hai et al. (2014) work with a special case of (5) in which u depends only on x_{1t}.

The most general formulation for allowing nonseparabilities in utility would simply assume that the agent has preferences over infinite consumption streams $\boldsymbol{x}$. One could then apply standard assumptions to ensure that these preferences can be represented by a utility function defined on the space of such consumption streams.

We have built additional structure into (4)–(5). First, we assume that preferences over intertemporal consumption streams are captured by a utility function that is the discounted sum of functions $\tilde{u}_t$, each of which depends upon only current and past consumption. Second, we split each function $\tilde{u}_t$ into two parts, one of which is a function only of current consumption (namely $u(x_{1t}, x_{2t})$) and one of which is a function of the current and all past consumption of good 2 (namely $\tilde{v}_t(x_{20}, \ldots, x_{2t})$). We can think of the function u as the counterpart of the typical utility function in a discounted-sum-of-utilities formulation such as (1), and the function $\tilde{v}_t$ as capturing nonseparabilities in preferences. Notice that $\tilde{v}_t$ and hence $\tilde{u}_t$ need the subscript t, because in different periods both will be functions of different arguments.

The decomposition of utility of a consumption stream into a discounted sum of instantaneous utility functions is familiar.[3] The transition from a period-t utility function of the form given in (3) to the form given on the right side of (4) separates the goods into ordinary goods and memorable goods. This is our basic distinction between goods. The foundations for this separation are given in Section 4.

2.2. Making memorable goods memorable

The role of the ordinary goods in (4)–(5) is straightforward. Without some additional structure, the intertemporal utility function given by (5) is capable of capturing a variety of nonseparabilities in intertemporal preferences. As we explain in Section 5.1, for example, depending on the functions involved, we might interpret this as a familiar model of habit formation or addiction. This section introduces additional assumptions that allow us to interpret the utility implications

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[3] Koopmans' (1960) axioms do not directly apply, because each x_{2t} appears also in future instantaneous utility values. Thus, the axioms need to be adjusted to apply to utility-equivalent bundles, where a change in x_{2t} is compensated by changes in future variables so that only the instantaneous utility at period t is affected. The argument then follows familiar lines but is tedious.

of memorable goods as indeed arising out of considerations having to do with the creation of future utility flows.

We would like the model to focus attention on the two aspects of utility highlighted in Section 1. First, the previous consumption of memorable goods $\{x_{20}, \ldots, x_{2t-1}\}$ enters the period-t utility function $\tilde{v}_t(x_{20}, \ldots, x_{2t})$ through flows of utility produced by the past consumption of such goods. One may enjoy fond memories of a vacation, wedding, or special night out long after they have occurred. Second, whether the new consumption of memorable goods produces future flows of utility depends on how this consumption compares to the consumer's customary consumption, with future utility flows emanating only from current consumption levels that are out of the ordinary. A dinner in the type of restaurant one visits weekly is unlikely to generate future utility, while a rare treat in a five-star restaurant may contribute to utility long after the evening is finished.

We capture these two forces in a parsimonious form by introducing two state variables, Υ_t and Λ_t. Intuitively, we interpret $\Upsilon_t \in \mathbb{R}_+$ as identifying the utility flows at t created by the past consumption of memorable goods and $\Lambda_t \in \mathbb{R}_+$ as identifying the customary level of consumption of the memorable good at time t. More precisely:

Assumption 1. There exists a function $\hat{v} : \mathbb{R}_+^2 \to \mathbb{R}$ and a constant $\nu \in (0, 1)$ such that

$$u(x_{1t}, x_{2t}) + \tilde{v}_t(x_{20}, \ldots, x_{2t}) = u(x_{1t}, x_{2t}) + \Upsilon_t + \hat{v}(x_{2t}, \Lambda_t)$$

$$= u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{t-1} \nu^{t-\tau} \hat{v}(x_{2\tau}, \Lambda_\tau) + \hat{v}(x_{2t}, \Lambda_t). \quad (6)$$

The first equality assumes that the function $\tilde{v}_t(x_{20}, \ldots, x_{2t})$ can be rewritten as a function of the arguments $(x_{2t}, \Upsilon_t, \Lambda_t)$, i.e. as a function of current consumption of the memorable good and the two state variables, and moreover is linear in the state variable Υ_t. The second equality gives meaning to the state variable Υ_t, assuming that Υ_t is a discounted sum of utilities generated by the consumption of memorable goods in the past, or

$$\Upsilon_t = \sum_{\tau=0}^{t-1} \nu^{t-\tau} \hat{v}(x_{2\tau}, \Lambda_\tau).$$

Notice that the continuing effects of the past consumption of memorable goods are separable from the effects of the current consumption of memorable goods.[4] When $\hat{v}(x_{2t}, \Lambda_t) > 0$, we say that the current consumption of good 2 is memorable, and such memorable consumption yields utility in the current and each subsequent period.

We next address the role of Λ in the function $\hat{v}$, in the process describing what (in our model) makes consumption memorable. The first two parts of our next assumption ensures that goods 1 and 2 are indeed "goods," in the sense that increased consumption increases the current utility u. In addition, we assume (in Assumption 2.3) that the consumption of x_2 generates additional utility in the current and future periods, in the form of a value $\hat{v}_t(x_{2t}, \Lambda_t) > 0$, if but only if the consumption x_{2t} is sufficiently large relative to the customary level of consumption of good 2. Assumption 2.4 indicates that the customary level of memorable-good consumption drifts in the direction of current consumption.

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[4] Hai et al. (2014) do not impose this assumption, though they do work with a discounted sum of utilities.

Assumption 2.

[2.1] The function u is strictly increasing.

[2.2] The function $\hat{v}$ is increasing in x_2 and decreasing in Λ, and is strictly increasing in x_2 and strictly decreasing in Λ whenever $\hat{v}(x_2, \Lambda) > 0$.

[2.3] There exists $\gamma > 1$ such that $x_2 \leq \gamma \Lambda \implies \hat{v}(x_2, \Lambda) = 0$.

[2.4] The customary consumption level Λ_t evolves according to, for $\lambda \in [0, 1]$,

$$\Lambda_t = \lambda \Lambda_{t-1} + (1 - \lambda) x_{2t-1}.$$

Assumptions 2.1 and 2.2 indicate that current utility is increasing in the consumption of both goods 1 and 2, but is decreasing in the customary consumption. The function U is monotonic in x_{1t} but need not be monotonic in x_{2t}, as an increase in the consumption of good 2 can have detrimental effects on future customary levels of utility that outweigh the salutary effects on current utility. Assumption 2.1 suffices to ensure that any optimal consumption plan must exhaust the consumer's budget.

2.3. Existence of optimal consumption plans

This section introduces the budget constraint and establishes that an optimal consumption plan exists.

We assume that the consumer has income I in each period, and that the interest factor at which she can borrow and save is equal to her discount factor, δ. Goods 1 and 2 are measured in units such that their prices are each 1.

Let Y_t denote the largest expenditure the consumer can make in period t, given that she can borrow any future income and spend any income that she has saved, after having paid for her previous consumption. Hence, we have

$$Y_0 = \frac{I}{1 - \delta}$$

and

$$Y_t = \left[Y_{t-1} - x_{1t-1} - x_{2t-1} \right] \frac{1}{\delta}.$$

The intertemporal budget constraint is that $Y_t \geq 0$, for all t. Intuitively, any expenditure larger than Y_{t-1} in period $t-1$ is impossible, being sufficiently large that the consumer's savings and discounted future income would not suffice to pay for it, which is equivalent to saying that consumption in period $t-1$ must be chosen so as to ensure $Y_t \geq 0$.

There is an upper bound on the value of Y_t, denoted by $\overline{Y}_t$, which the consumer achieves by spending nothing on consumption in periods $\{0, \ldots, t-1\}$, given by

$$\overline{Y}_t = \frac{I}{\delta^t} + \frac{I}{\delta^{t-1}} + \ldots + \frac{I}{\delta^2} + \frac{I}{\delta} + \frac{I}{1 - \delta} = \frac{I}{\delta^t (1 - \delta)}.$$

We thus have $Y_t \in [0, \overline{Y}_t]$. Notice that $\overline{Y}_t$ grows arbitrarily large as does t.

The consumer's objective is then to choose a consumption plan $\{(x_{1t}, x_{2t})\}_{t=0}^{\infty}$ to solve the constrained utility maximization problem given by:

$$\max \sum_{t=0}^{\infty} \delta^t \left[u(x_{1t}, x_{2t}) + \Upsilon_t + \hat{v}(x_{2t}, \Lambda_t) \right] \tag{7}$$

$$\text{s.t.} \quad Y_t = [Y_{t-1} - x_{1t-1} - x_{2t-1}]^{\frac{1}{\delta}} \geq 0 \tag{8}$$

$$\Upsilon_t = \nu(\Upsilon_{t-1} + \hat{v}(x_{2t-1}, \Lambda_{t-1})) \tag{9}$$

$$\Lambda_t = \lambda \Lambda_{t-1} + (1-\lambda)x_{2t-1} \tag{10}$$

$$(x_{1t}, x_{2t}) \in \mathbb{R}_+^2, \tag{11}$$

given initial values $(Y_0, \Upsilon_0, \Lambda_0)$.

Without further assumptions, this problem need not have a solution. We accordingly invoke the following. There is some scope for weakening these assumptions—for example, differentiability is not essential, but it is convenient.

Assumption 3.

[3.1] The utility function u is continuously differentiable on $\mathbb{R}_+^2$. The utility function $\hat{v}$ is continuously differentiable when it is positive.

[3.2] The utility function u is strictly concave. The utility function $\hat{v}$ is strictly concave in x_2 on that part of its domain in which it is positive.

[3.3] $\lim_{x_1 \to \infty} u_1(x_1, x_2) = \lim_{x_2 \to \infty} u_2(x_1, x_2) = 0$.

[3.4] For any sequence $\{(x_{1\tau}, x_{2\tau})\}_{\tau=0}^{\infty}$ with $\lim_{\tau \to \infty} x_{1\tau} = \lim_{\tau \to \infty} x_{2\tau} = \infty$, we have either $\lim_{\tau \to \infty} \frac{du(x_{1\tau}, x_{2\tau})}{dx_{1\tau}} = 0$ or $\lim_{\tau \to \infty} \frac{du(x_{1\tau}, x_{2\tau})}{dx_{2\tau}} = 0$.

[3.5] For all sequences $\{x_{2\tau}\}_{\tau=0}^{\infty}$ with $\lim_{\tau \to \infty} x_{2\tau} = \infty$, we have $\lim_{\tau \to \infty} \frac{d\hat{v}(x_{2\tau}, \Lambda)}{dx_{2\tau}} = 0$ uniformly in Λ.

Assumption 3.1 imposes familiar smoothness conditions. Assumption 3.2 gives rise to pressure to smooth consumption. Assumption 3.2 ensures that marginal utilities are decreasing, and Assumptions 3.3–3.5 address limiting marginal utilities. Assumptions 3.3 and 3.4 indicate that arbitrarily large values of consumption ensure that at least one of the marginal utilities of u is arbitrarily small, and the final assumption imposes a similar requirement for the function $\hat{v}$. The uniform convergence requirement in Assumption 3.5 may appear to be stringent. However, we will typically think of $\frac{d^2\hat{v}(x_{2\tau}, \Lambda_\tau)}{dx_{2\tau}d\Lambda_t} \leq 0$. It will then suffice for the condition in Assumption 3.5 to hold when $\Lambda = 0$.

The consumer's intertemporal maximization problem has a solution:

Proposition 1. *Let Assumptions 1–3 hold. Then there exists a consumption plan $x^* : \mathbb{R}_+^3 \to \mathbb{R}_+^2$, identifying values of (x_{1t}, x_{2t}) in each period t as a function of $(Y_t, \Upsilon_t, \Lambda_t)$, that solves (7)–(11). Moreover, there exists a continuous value function $V : \mathbb{R}_+^3 \to \mathbb{R}$ such that the maximization problem can be written as*

$$V(Y_t, \Upsilon_t, \Lambda_t) = \max_{x_{1t}, x_{2t}} u(x_{1t}, x_{2t}) + \Upsilon_t + \hat{v}(x_{2t}, \Lambda_t) + \delta V(Y_{t+1}, \Upsilon_{t+1}, \Lambda_{t+1})$$

subject to the constraints given in (8)–(11), given initial values $(Y_0, \Upsilon_0, \Lambda_0)$.

Given its additive form in (6), the variable Υ_0 affects the value of V, but not the optimal continuation strategy.

The remainder of this section comprises the proof of Proposition 1. Our first step in establishing this result is to show that there is no loss of generality in replacing (11) with the constraint

$$(x_{1t}, x_{2t}) \in X, \tag{12}$$

for some compact $X \subset \mathbb{R}_+^2$.

Lemma 2. *Let Assumptions 1–3 hold. Then there exists a finite $\overline{x}$ such that any consumption plan featuring a period t in which $x_{it} > \overline{x}$ for either $i = 1, 2$ is dominated by a consumption plan in which $x_{it} \leq \overline{x}$ for $i = 1, 2$ and all t.*

The proof, contained in Section 6.1, first notes that the consumer's marginal utilities in the first period are bounded below, even if the consumer concentrates all of her consumption in the first period. We then use Assumptions 3.2–3.5 to argue that any unbounded consumption plan must eventually feature a marginal utility smaller than the bound from the first period. The consumer can then increase utility by shifting consumption to the first period, ensuring that the plan in question is not optimal. The intertemporal links created by memorable goods introduce only slight complications in this otherwise quite familiar line of argument.

We can thus take the set X in (12) to be the set $[0, \overline{x}]^2$, and our task is then to show that the problem given by (7)–(10) and (12) has a solution. For this, we need only note that Assumption 3, along with the compactness of X, ensure that the hypotheses of Theorem 12.19 of Sundaram (1996) are satisfied, which delivers the result.[5]

2.4. Simplifications

This section introduces two simplifications. The first simplification is purely a matter of notation. It is helpful to rearrange the intertemporal objective as follows:

$$\sum_{t=0}^{\infty} \delta^t \left[u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{t-1} v^{t-\tau} \hat{v}(x_{2\tau}, \Lambda_\tau) + \hat{v}(x_{2t}, \Lambda_t) \right]$$

$$= \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t}) + \sum_{t=0}^{\infty} \delta^t \sum_{\tau=0}^{t} v^{t-\tau} \hat{v}(x_{2\tau}, \Lambda_\tau)$$

$$= \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{\infty} \sum_{t=\tau}^{\infty} \delta^t v^{t-\tau} \hat{v}(x_{2\tau}, \Lambda_\tau)$$

$$= \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{\infty} \delta^\tau \frac{1}{1 - \delta v} \hat{v}(x_{2\tau}, \Lambda_\tau)$$

$$= \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{\infty} \delta^\tau v(x_{2\tau}, \Lambda_\tau)$$

$$= \sum_{t=0}^{\infty} \delta^t [u(x_{1t}, x_{2t}) + v(x_{2t}, \Lambda_t)].$$

The first expression is taken from (7) and the definition of Υ_t. The first equality distributes the initial summation. The next equality interchanges the order of summation in the double sum.

The next equality then simplifies the second sum in the double sum. The following equality introduces the function $v = \frac{1}{1-\delta v}\hat{v}$. The final equality collects the terms in a single summation.

This formulation has the advantage of focusing attention on the periods in which the consumption of memorable goods gives rise to lasting flows of utility, while clearing from view (but not neglecting) the subsequent accounting for these flows. The function v is proportional to $\hat{v}$, and hence inherits the properties of $\hat{v}$ given in Assumptions 2–4.

The following assumption is used in Section 3.4, but not elsewhere in the paper:

Assumption 4. The functions u and $\hat{v}$ are homogeneous of degree $\alpha < 1$.

3. Optimal consumption plans

We now turn to a characterization of optimal consumption plans in the presence of memorable goods. We examine an infinite-horizon model in which consumption would be perfectly smoothed in the absence of memorable consumption, with a particular emphasis on whether the presence of memorable goods disrupts this smoothing.[6]

3.1. Benchmark: memoryless consumption

To provide a comparison, let us recall the familiar special case in which there is no memorable consumption. We assume that the relevant parts of Assumptions 3 hold, so that u is differentiable, increasing and concave.

The consumer's problem is

$$\max_{\{x_{1t},x_{2t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t})$$

$$\text{s.t.} \quad \sum_{\tau=0}^{\infty} \delta^t (x_{1t} + x_{2t}) = Y_0.$$

The first-order conditions for an interior solution for this maximization problem call for the marginal utilities to be equalized across goods and across periods. The concavity of u then ensures that we have an interior solution exhibiting perfect consumption smoothing. There exist quantities x_1^* and x_2^* with $x_1^* + x_2^* = I$ such that $x_{1t} = x_1^*$ and $x_{2t} = x_2^*$ for all t.

3.2. No acclimatization

We first consider a special case, namely that in which $\lambda = 1$, so that there is no acclimatization of the customary level of consumption of the memorable good in response to current consumption. This case is particularly easy to characterize—either the consumer will perfectly smooth consumption, or the consumer's indirect utility function will be effectively linear over the relevant range. In the latter case, consumption may not be perfectly smoothed, but this lack

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[6] We cannot expect perfect consumption smoothing in the presence of memorable consumption and a *finite*, deterministic lifetime. Memorable consumption generated in the first period of a two-period model is enjoyed in both periods, while memorable consumption generated in the final period can necessarily be enjoyed only in that period. This provides a natural tendency to front-load memorable consumption. We would accordingly expect young people to spend a relatively larger share of their income on weddings than do senior citizens.
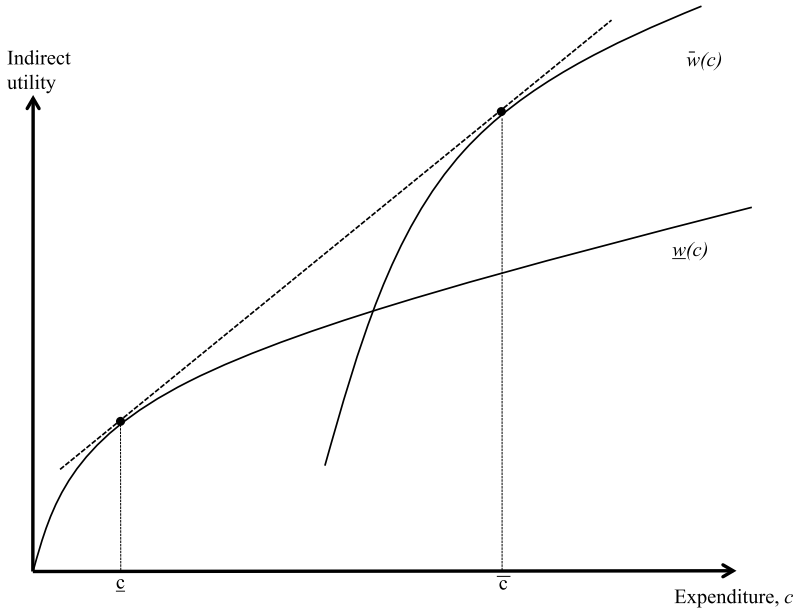
Fig. 1. Indirect utility functions $\underline{w}$ and $\overline{w}$ for the case of a perfectly persistent customary level Λ. Expenditure on consumption within a period is denoted by c, and $\underline{w}(c)$ and $\overline{w}(c)$ denote the maximal utility in that period given consumption c and given that memorable consumption does not ($\underline{w}$) or does ($\overline{w}$) occur.

of smoothing is inconsequential. The path of consumption will be drawn from a set of optimal consumption paths, with the linearity of the indirect utility function ensuring that all such plans have equivalent utilities. These results show that the tendency of the customary consumption level to drift toward actual consumption plays an essential role in the link between memorable consumption and lumpy consumption.

Suppose that the customary level of consumption is perfectly persistent, so that $\Lambda_t = \Lambda$ for all t, regardless of history. The utility function is constant across periods in this case, and is given by

$$u(x_1, x_2)$$

when $x_2 \leq \gamma \Lambda$ and is given by

$$u(x_1, x_2) + v(x_2, \Lambda)$$

when $x_2 \geq \gamma \Lambda$. Let c_t be the total amount spent on consumption in period t. Then we can define single-period indirect utility functions $\underline{w}(c) = \max_{x_1 + x_2 \leq c} u(x_1, x_2)$ for the case in which memorable consumption does not occur in the period in question and $\overline{w}(c) = \max_{x_1 + x_2 \leq c} u(x_1, x_2) + v(x_2, \Lambda)$ for the case in which memorable consumption occurs. Given the stationarity of Λ, we can write these solely as a function of c. In particular, no intertemporal considerations are involved in deriving these functions. It is straightforward to verify that both functions are concave, and that $\overline{w}$ is more steeply sloped. We illustrate the indirect utility functions in Fig. 1. Let $w(c) = \max\{\underline{w}(c), \overline{w}(c)\}$. We can use these indirect utility functions to characterize the optimal consumption plan in this case.

Let $\hat{w}$ be the smallest concave function larger than $w(c)$. Then $\hat{w}$ is given by the upper envelope of the utility functions $\underline{w}$, $\overline{w}$ and the dashed tangent in [Fig. 1](#), and $\underline{c}$ and $\overline{c}$ are the points of intersection of the tangent and the functions $\underline{w}$ and $\overline{w}$.

Our strategy is now as follows. We derive the optimal consumption plan for the function $\hat{w}$. This is relatively straightforward, since we have a utility function that is fixed across periods and concave. The details of this plan will depend on the level of income. For each level of income, we have an optimal consumption plan and an induced sequence of utilities, given utility function $\hat{w}$. We then show that either the original induced sequence of utilities, or in some cases a different but feasible sequence, gives the same total (discounted) utility under utility function w. This ensures that the resulting plan is optimal for w.

The first observation is that since $\hat{w}$ is concave, it is always optimal to equalize consumption across periods. Let

$$\frac{\hat{c}}{1-\delta} = Y_0.$$

Then $\hat{c}$ is the unique consumption level consistent with the consumer's income and consuming the same amount in each period. This now leads to three cases.

If $\hat{c} \leq \underline{c}$, then consuming $\hat{c}$ in each period under utility function w gives $w(\hat{c}) = \hat{w}(\hat{c})$, and hence we have an optimal consumption plan for utility w. In this case, the consumer's income is too low to make it worth engaging in memorable consumption. The fixed customary level Λ ensures that, even though the consumer never undertakes memorable consumption, the customary level never falls to a point that would make memorable consumption worthwhile.

If $\hat{c} \geq \overline{c}$, then consuming $\hat{c}$ in each period under utility function w gives $w(\hat{c}) = \hat{w}(\hat{c})$, and hence we have an optimal consumption plan for utility w. In this case, the consumer's income is sufficiently large that the consumer enjoys memorable consumption in every period, with the customary level Λ never increasing as a result.

Suppose $\hat{c} \in (\underline{c}, \overline{c})$. Now we do not have $w(\hat{c}) = \hat{w}(\hat{c})$, since $\hat{w}(\hat{c})$ falls on the line segment that "concavifies" w. However, this line segment is linear. As a result, we can replace the sequence that consumes $\hat{c}$ in every period with a sequence that consumes $\underline{c}$ in some periods and $\overline{c}$ in others. The latter is feasible, and we argue that when this sequence is evaluated with the utility function w, it gives the same discounted utility sum as does the constant sequence $\hat{c}$ evaluated under the utility function $\hat{w}$. As we have argued, this suffices for the result.

In particular, given two periods t and $t' > t$, the hypothetical utility $\hat{w}(\hat{c})$ is indifferent over pairs $(c_t, c_{t'})$ that satisfy $\underline{c} \leq c_t$, $c_{t'} \leq \overline{c}$ and $c_t + \delta^{t'-t} c_{t'} = \hat{c}(1 + \delta^{t'-t})$. This in turn means that the consumer is indifferent over variations in c_t and c'_t that satisfy

$$\frac{dc_{t'}}{dc_t} = -\frac{1}{\delta^{t'-t}},$$

which is precisely the rate at which these two can be traded off in order to preserve feasibility. This in turn implies that *any* feasible consumption plan that features only $\underline{c}$ and $\overline{c}$ is optimal—when this sequence is evaluated with the utility function w, it gives the same discounted utility sum as does the constant sequence $\hat{c}$ evaluated under the utility function $\hat{w}$. At one extreme in the collection of such sequences is a plan that first consumes only $\overline{c}$, until switching to the perpetual consumption of $\underline{c}$. This is a consumer who first binges on memorable consumption, and then forsakes it entirely. At the other extreme is a plan that first consumes only $\underline{c}$, until switching to the perpetual consumption of $\overline{c}$. This is a consumer who delays gratification. We have thus established the following:

Proposition 3. *Let Assumptions 1–3 hold, and let $\lambda = 1$, so that the level of customary memorable-good consumption shows no acclimatization. Then either*

  (i) *the consumer never attains memorable consumption (if Y_0 is sufficiently small);*
 (ii) *the consumer always enjoys memorable consumption (if Y_0 is sufficiently large); or*
(iii) *there will exist expenditure levels $\underline{c} < \overline{c}$ such that any consumption plan that satisfies the budget constraint and exhibits only the consumption levels $\underline{c}$ and $\overline{c}$ is optimal.*

The third case includes an infinite number of consumption plans that distribute $\underline{c}$ and $\overline{c}$ seemingly arbitrarily across periods, with the only constraint being that the resulting consumption plan exhausts the consumer's budget.

3.3. Concave–convex–concave utility functions

The utility functions shown in Fig. 1 exhibit the concave–convex–concave shape discussed by Friedman and Savage. If we assumed that the consumer uses the same function w for utility maximization under risk, then we would apparently have an explanation for the simultaneous purchases of insurance and lotteries. Intuitively, the bonus provided by the ability to translate a large payoff into memorable consumption can make an actuarially fair lottery attractive. However, in the presence of an infinite horizon, this consumer would have no interest in buying a lottery. No fair (or worse than fair) lottery can offer the consumer the possibility of memorable consumption on terms better than the consumer can achieve by shifting consumption across periods.

We can gain some insight into this by examining a two-period example. We maintain in the example the assumption that there is no acclimatization, so that the customary level is fixed at some value Λ. The agent's objective is

$$\max_{x_{10}, x_{20}, x_{11}, x_{21}} \left\{ u(x_{10}, x_{20}) + \hat{v}(x_{20}, \Lambda) + \delta \left[u(x_{11}, x_{21}) + v \hat{v}(x_{20}, \Lambda) + \hat{v}(x_{21}, \Lambda) \right] \right\}, \quad (13)$$

subject to the budget constraint

$$x_{10} + x_{20} + \delta(x_{11} + x_{21}) = Y_0.$$

We make the example more concrete by assuming

$$u(x_1, x_2) = \frac{x_1^\alpha}{\alpha} + \frac{x_2^\alpha}{\alpha} \tag{14}$$

$$\hat{v}(x_2, \Lambda) = \xi \max \left\{ 0, \frac{x_2^\alpha}{\alpha} - \gamma \frac{\Lambda^\alpha}{\alpha} \right\} \tag{15}$$

for $\xi > 0$ and $\gamma > 1$. We assume $\alpha \in (0, 1)$, with $\alpha < 1$ ensuring that the functions are concave, and $\alpha > 0$ ensuring that utilities are nonnegative and hence 0 is a relevant comparison for the maximum in the specification of v.

Because it contains two maximum operators, in the two functions $\hat{v}$, the agent's objective given by (13) is neither concave nor differentiable. In response, we separately examine three objective functions for the agent, corresponding to three ways the maximum operators might be resolved (explaining below why the fourth possibility is irrelevant). For each case we have a concave and differentiable function, allowing us to use first-order conditions to find their (necessarily interior) solutions. We derive the indirect utility function for each case. We then note that the indirect utility function corresponding to (13) is given by the upper envelope of these three indirect utility functions, allowing us to make the inferences we need about the solution to (13).

In the first case, the consumer engages in no memorable consumption. The consumer's utility is then given by

$$\frac{x_{10}^{\alpha}}{\alpha} + \frac{x_{20}^{\alpha}}{\alpha} + \delta \left[\frac{x_{11}^{\alpha}}{\alpha} + \frac{x_{21}^{\alpha}}{\alpha} \right].$$

The first-order conditions for utility maximization give $x_{10}^{\alpha-1} = x_{20}^{\alpha-1} = x_{11}^{\alpha-1} = x_{21}^{\alpha-1}$, and we can solve for

$$x_{10} = x_{20} = x_{11} = x_{21} = \frac{Y_0}{2 + 2\delta}.$$

Here, we have perfect smoothing of the consumption of both goods, as we would obtain with the type of utility function given by (1). Let $\underline{V}(Y_0)$ be the indirect utility function, given the constraint that the consumer never generates memorable utility (and holding all other variables, most notably Λ, fixed). The envelope theorem then gives[7]

$$\frac{d\underline{V}}{dY_0} = \left(\frac{Y_0}{2 + 2\delta} \right)^{\alpha-1}.$$

Notice that $\underline{V}(Y_0)$ is concave.

In the second consumption plan, the consumer engages in memorable consumption in only a single period. It is immediate that the consumer will do so in period 0. It is here that we see implications of the durability of the utility flows generated by memorable consumption. Given that the customary level of memorable-good consumption is fixed, and given that memorable consumption occurs in only one period, then that period will be the first, so as to take advantage of the additional utility generated in the second period. The consumer's utility is then

$$\frac{x_{10}^{\alpha}}{\alpha} + \frac{x_{20}^{\alpha}}{\alpha} + \left(\frac{x_{20}^{\alpha}}{\alpha} - \gamma \frac{\Lambda_0^{\alpha}}{\alpha} \right) \xi (1 + \delta\nu) + \delta \left[\frac{x_{11}^{\alpha}}{\alpha} + \frac{x_{21}^{\alpha}}{\alpha} \right].$$

The first-order conditions for utility maximization give $x_{10}^{\alpha-1} = x_{20}^{\alpha-1}[1 + \xi + \xi\nu\delta] = x_{11}^{\alpha-1} = x_{21}^{\alpha-1}$, and we can solve for

$$x_{10} = x_{11} = x_{21} = \frac{\theta Y_0}{\theta(1 + 2\delta) + 1} \qquad x_{20} = \frac{Y_0}{\theta(1 + 2\delta) + 1},$$

where

$$\theta = [1 + \xi + \xi\nu\delta]^{\frac{1}{\alpha-1}} \in (0, 1).$$

Here, consumption of the first good is perfectly smoothed, but the consumption of the memorable good (and hence total consumption) is decreasing. Compared to the previous case, the memorable consumption in period 0 prompts an increase in x_{20}, because the memorable consumption increases the marginal utility of x_{20}, and prompts a corresponding decrease in all other variables so as to preserve the budget constraint. Let $V(Y_0)$ be the indirect utility function given the constraint that the consumer engages in memorable consumption in period 0 (only). The envelope theorem then gives[8]

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[7] The domain of this indirect utility function is restricted to income levels Y_0 sufficiently small that no memorable utility is generated when consumption is perfectly smooth.

[8] The domain of this indirect utility function is restricted to income levels Y_0 sufficiently large that memorable consumption is possible in the first period, but not so large that the consumption bundle solving the resulting first-order conditions would also give memorable consumption in the second period.

$$\frac{dV}{dY_0} = \left(\frac{\theta Y_0}{\theta(1 + 2\delta) + 1}\right)^{\alpha - 1}.$$

Notice that $V(Y_0)$ is concave and is steeper than $\underline{V}$.

Now suppose that the consumer engages in memorable consumption in both periods. The consumer's utility is then

$$\frac{x_{10}^\alpha}{\alpha} + \frac{x_{20}^\alpha}{\alpha} + \left(\frac{x_{20}^\alpha}{\alpha} - \gamma \frac{\Lambda_0^\alpha}{\alpha}\right)\xi(1 + \delta\nu) + \delta\left[\frac{x_{11}^\alpha}{\alpha} + \frac{x_{21}^\alpha}{\alpha} + \left(\frac{x_{21}^\alpha}{\alpha} - \gamma \frac{\Lambda_0^\alpha}{\alpha}\right)\xi\right].$$

The first-order conditions for utility maximization give $x_{10}^{\alpha-1} = x_{20}^{\alpha-1}[1 + \xi + \xi\nu\delta] = x_{11}^{\alpha-1} = x_{21}^{\alpha-1}(1 + \xi)$, and we can solve for

$$x_{10} = x_{11} = \frac{\theta\phi Y_0}{(1 + \delta)\theta\phi + \phi + \delta\theta}, \qquad x_{20} = \frac{\phi Y_0}{(1 + \delta)\theta\phi + \phi + \delta\theta}$$

$$x_{21} = \frac{\theta Y_0}{(1 + \delta)\theta\phi + \phi + \delta\theta},$$

where θ is as before and

$$\phi = [1 + \xi]^{\frac{1}{\alpha-1}} \in (0, 1).$$

Once again, the consumption of good one is perfectly smoothed, but consumption of the memorable good (and hence total consumption) is increasing. This pattern is consistent with the preference for an increasing consumption path that could arise out of a utility function of the form given by (2), though for somewhat different reasons. Compared to the previous case, the additional memorable consumption in period 1 prompts an additional decrease in $x_{10} = x_{11}$, since the presence of memorable consumption in both periods increases yet further the marginal utility gains from doing so. Let $\overline{V}(Y_0)$ be the indirect utility function given the constraint that the consumer generate memorable utility in both periods. The envelope theorem then gives[9]

$$\frac{d\overline{V}}{dY_0} = \left(\frac{\theta\phi Y_0}{(1 + \delta)\theta\phi + \phi + \delta\theta}\right)^{\alpha - 1}.$$

Notice that $\overline{V}(Y_0)$ is concave and is steeper than V.

Fig. 2 illustrates the indirect utility functions for this example. In this case we find a utility function that is concave in some regions but not others. An agent described by this utility function could well be willing to both purchase insurance and gamble. However, the willingness to do so in general rests on some "friction" in the consumer's ability to transfer consumption across periods. One such friction is a finite lifetime, appearing in Friedman and Savage (1948) in the form of a single-period horizon. Were the agent Friedman and Savage analyze infinitely-lived, the ability to transfer consumption across periods would obviate the gains from gambling.

3.4. Acclimatization

We now turn to the case $\lambda \in [0, 1)$, so that acclimatization occurs, including the special case of $\lambda = 0$, or immediate acclimatization. We maintain Assumptions 1–4 throughout.

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[9] The domain of this indirect utility function is restricted to income levels Y_0 sufficiently large that memorable consumption is possible in both periods.
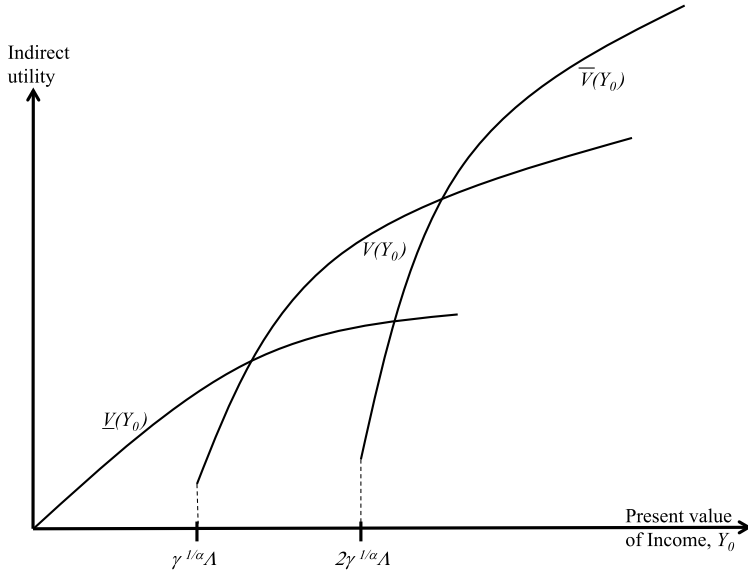
Fig. 2. Indirect utility functions for the first specification of the example, featuring no acclimatization. The three indirect utility functions correspond to the case in which there is no memorable consumption ($\underline{V}$), there is memorable consumption only in the first period (V), and memorable consumption in both periods ($\overline{V}$).

We first argue that memorable consumption is not in general a transient phenomenon—the consumer avails herself of memorable utility infinitely often. Let $x_1^*(Y_0)$ and $x_2^*(Y_0)$ be the consumption quantities that would be optimal, in period 0 and every subsequent period, if we assumed that the function v is identically equal to zero. The stock Λ is irrelevant in this case, and these quantities can be written solely as a function of Y_0. As we have noted in Section 3.1, these quantities will be constant across time periods, and so no time subscripts are needed. We are of course interested in the case in which v is nonzero, and $x_1^*(Y_0)$ and $x_2^*(Y_0)$ will be useful for the analysis of this case.

Definition 1. We say that memorable consumption is *felicitous* if, when $\Lambda_0 = x_2^*(Y_0)$, the optimal consumption plan calls for memorable consumption at least once, and yields a utility strictly higher than never engaging in memorable consumption.

The interpretation of the condition that $\Lambda_0 = x_2^*(Y_0)$ is that the consumer's initial customary level of consumption of the memorable good matches the level of consumption that would be relevant if memorable consumption never occurred. Memorable consumption is felicitous if, in this circumstance, the consumer would find it optimal to at least sometimes engage in memorable consumption.

Assumption 4 requires that the functions u and $\hat{v}$ be homogeneous of degree $\alpha < 1$. Nothing to this point needed this assumption, but we use it in what follows. An implication is that if $\{x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is an optimal consumption plan given (Y_0, Λ_0), then $\{\zeta x_{1t}^*, \zeta x_{2t}^*\}_{t=0}^{\infty}$ is an optimal consumption plan given $(\zeta Y_0, \zeta \Lambda_0)$ for any $\zeta > 0$. Hence, the question of whether memorable consumption is felicitous does not depend on Y_0, ensuring that the property of felicity is well defined.

Felicity is defined in terms of endogenous objects. However, we can easily find (less insightful) conditions on primitives ensuring that memorable consumption is felicitous. For example, let $\{v_n\}$ be a sequence of functions, satisfying the properties placed on v by Assumptions 2–4, and suppose that the sequence is pointwise increasing and pointwise unbounded for any argument (x_2, Λ_0) with $x_2 > \Lambda_0$.[10] Then there exists a value N such that for all $n \geq N$, letting $v = v_n$ ensures that memorable utility is felicitous. Hence, memorable consumption is felicitous if the technology for generating future utility flows is sufficiently productive.

If memorable consumption is felicitous, then not only does an optimal consumption plan exhibit memorable consumption, but it does so infinitely often:

Proposition 4. *Let $\lambda \in [0, 1)$, let Assumptions 1–4 hold, and let memorable consumption be felicitous. Then in an optimal consumption plan, memorable consumption occurs infinitely often.*

The proof, in Section 6.2, begins by supposing that memorable consumption occurs at most finitely many times. After the last incidence of memorable consumption, the problem of maximizing the continuation utility is equivalent to the memoryless utility maximization problem considered in Section 3.1. Then, we note that the continuation consumption plan must exhibit the consumption of some bundle (x_1^*, x_2^*) in every period, and hence $\lim_{t \to \infty} \Lambda_t = x_2^*$. But then the homogeneity imposed by Assumption 4 ensures that the optimal continuation consumption plan must approach a limit that is proportional to the original plan, which combined with the assumption that memorable consumption is felicitous ensures that memorable consumption once again occurs.

In the setting of Section 3.2, with no acclimatization, the initial customary level Λ_0 plays a key role in determining whether the optimal consumption plan exhibits memorable consumption. The optimal consumption plan calls for memorable consumption if Λ_0 is sufficiently small and does not if Λ_0 is sufficiently large. In the current setting, the initial level Λ_0 plays no role in determining whether memorable consumption is felicitous, and plays no role in determining whether optimal consumption plans exhibit memorable consumption (when memorable consumption is felicitous). Notice that if memorable consumption is *not* felicitous, then the optimal consumption plan will exhibit memorable consumption if Λ_0 is sufficiently small, but need not do so infinitely often.

On the other side, and once again in contrast to the case of no acclimatization, even if memorable consumption sometimes occurs, it cannot do so in every period:

Proposition 5. *Let $\lambda \in [0, 1)$ and let Assumptions 1–4 hold. Then for every T, there exists a period $t > T$ in which memorable consumption does not occur.*

The proof is straightforward, and so we offer only a sketch of the argument. If memorable consumption occurs in period t, we have

$$
\begin{aligned}
\Lambda_{t+1} &= \lambda \Lambda_t + (1 - \lambda) x_{2t} \\
&\geq \lambda \Lambda_t + (1 - \lambda) \gamma \Lambda_t \\
&= [\lambda + (1 - \lambda)\gamma] \Lambda_t.
\end{aligned}
$$

---

[10] Notice that along the sequence $\{v_n\}$, the factor γ determining the extent to which the consumption of memorable goods must exceed the customary level in order to generate memorable utility is shrinking.

Hence, if there exists a time after which memorable consumption occurs in every period, then the customary level Λ_t must grow without bound (since $\lambda + (1 - \lambda)\gamma > 1$), as must the consumption level x_{2t}. We have already seen, as the essential lemma used in proving Proposition 1, that optimal consumption plans are bounded.

Paired with Proposition 4, this result indicates that an optimal consumption plan must involve memorable consumption infinitely often (when memorable consumption is felicitous), but must intersperse such consumption with periods in which no memorable consumption occurs. A key feature of the latter is that they allow the customary level to decline to the point that memorable consumption is again optimal. Consumption thus switches back and forth between periods involving memorable consumption and periods in which the customary level of memorable good consumption is allowed to decline.

We can say something about the intervals in which memorable consumption occurs. Section 6.3 proves:

Proposition 6. *Let* $\lambda \in [0, 1)$ *and let* Assumptions 1–4 *hold, and consider an interior solution to the consumer's maximization problem. Suppose that the optimal consumption plan involves memorable consumption in period* t' *and* $t'' > t'$, *but not in the intervening periods. Then over the course of the periods* $(t' + 1, \ldots, t'' - 1)$, *the marginal utility* $u_1(x_{1t}, x_{2t})$ *of good 1 remains constant, while the direct marginal utility* $u_2(x_{1t}, x_{2t})$ *of good 2 increases.*

It is a standard result that as long as $\delta > 0$, the marginal utility of good 1 is optimally equalized across periods. If not, the discounted sum of utilities could be increased by shifting the consumption of good 1 from low-marginal-utility to high-marginal-utility periods. Much the same intuition holds for good 2. In this case, however, the relevant marginal utility considerations involve not only the immediate marginal utility in the period of consumption, but also the marginal effect on the customary level of good-2 consumption in each future period in which memorable consumption occurs. Indeed, the optimality conditions for good 2 trade off the immediate utility-enhancing effects of increased consumption against the utility-decreasing effects of higher future customary levels. The difference between periods t and $t + 1$ (with $t' < t < t + 1 < t''$) is that in the case of the latter, these future impacts on customary levels are stronger and closer. This makes it all the more important to attenuate these future effects in period $t + 1$, leading to a higher marginal utility.

These forces are especially convenient to illustrate when the switching back and forth between periods in which memorable consumption occurs and intervals without memorable consumption induces a perfect cycle. Such a cycle is characterized by a number n, with memorable consumption occurring every n periods, and with the state variables (Y, Υ, Λ) taking on the same values at the beginning of every nth period. Let a sequence of such periods be numbered $1, 2, \ldots, n$ with memorable consumption in period 1. We can let the consumption levels in these n periods be denoted by $((x_{11}, x_{12}), \ldots, (x_{1n}, x_{2n}))$. Then there exists a wealth level Y, intuitively giving the current discounted value of expenditures over the n periods beginning with memorable utility generation, and a level of customary consumption Λ, giving the customary consumption at the beginning of each period in which memorable consumption occurs, such that the optimal consumption plan must satisfy:

$$\max_{((x_{11},x_{21}),\ldots,(x_{1n},x_{2n}))} u(x_{11}, x_{21}) + v(x_{21}, \Lambda_1) + \delta u(x_{12}, x_{22}) + \ldots + \delta^{n-1} u(x_{1n}, x_{2n})$$

$$\text{s.t.} \quad Y = x_{11} + x_{21} + \delta(x_{12} + x_{22}) + \delta^2(x_{13} + x_{23}) + \ldots + \delta^{n-1}(x_{1n} + x_{2n})$$

$$\Lambda = \lambda^n \Lambda + \lambda^{n-1}(1-\lambda)x_{21} + \lambda^{n-2}(1-\lambda)x_{22} +$$
$$\ldots + \lambda(1-\lambda)x_{2n-1} + (1-\lambda)x_{2n}.$$

This maximization problem says nothing about what determines n, Y and Λ, but nonetheless the optimal stationary policy must solve this maximization problem.

Letting ζ be the multiplier on the first constraint and ψ the multiplier on the second, we can formulate the first-order conditions for an interior solution as

$$u_1(x_{11}, x_{21}) + \zeta = 0$$
$$u_1(x_{12}, x_{22}) + \zeta = 0$$
$$\vdots$$
$$u_1(x_{1n-1}, x_{2n-1}) + \zeta = 0$$
$$u_1(x_{1n}, x_{2n}) + \zeta = 0$$

and

$$u_2(x_{11}, x_{21}) + v_1(x_{21}, \Lambda_1) + \zeta + \psi\lambda^{n-1}(1-\lambda) = 0$$
$$u_2(x_{12}, x_{22}) + \zeta + \psi\lambda^{n-2}\delta^{-1}(1-\lambda) = 0$$
$$\vdots$$
$$u_2(x_{1n-1}, x_{2n-1}) + \zeta + \psi\lambda\delta^{-(n-2)}(1-\lambda) = 0$$
$$u_2(x_{1n}, x_{2n}) + \zeta + \psi\delta^{-(n-1)}(1-\lambda) = 0.$$

The marginal utility of good 1 is equalized across periods in such a solution. This reflects a standard consumption-smoothing argument. The marginal utility of good 2 increases as the next bout of memorable consumption draws near. Reducing the consumption of good 2 reduces the customary level against which the next instance of memorable consumption is measured. The closer is the next instance of memorable consumption, the more valuable is this reduction, and hence the larger the marginal utility of good 2. This gives us a consumption pattern for good 2 that peaks with the incidence of memorable consumption, then takes a drop, and then declines until the next occurrence of memorable consumption.

If the utility function exhibits a positive cross partial derivative, then the consumption pattern for memorable goods will spill over into a similarly cyclic behavior for the consumption of good 1. Hence, memorable consumption can induce cycles in the consumption of goods that are inherently nonmemorable.[11]

4. Foundations

A key feature of our model is that past consumption can affect utility through acclimatization and through future utility flows. This calls for a model in which the utility function at time t depends not only on current values of consumption x_{1t} and x_{2t}, but also on their past values. However, such a function allows for a wide variety of history-dependent utility functions. In order to focus on the effects that are of interest to us, we suggested the instantaneous utility function

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[11] As noted in Section 1.3, this effect does not arise in Hai et al. (2014).

given in (4), which is the sum of two functions. One function depends only on the goods consumed at present, x_{1t} and x_{2t}, capturing the standard, non-memory-related utility, and the other function depends only on past and current values of x_2, capturing the effects of acclimatization and memorable consumption.

It is not entirely clear what we assume by this functional form. According to the classical notion of separability the utility function is the sum of two (or more) functions, each of which has a disjoint set of variables. But what is assumed by a summation of two functions whose sets of variables are *not* disjoint? Clearly, not every function can be so written. Yet, such functions do not satisfy the conditions of separability.

The purpose of this section is to axiomatize a functional form as in (4). In general, axioms on presumably observed preferences (interpreted as the instantaneous preferences at time t) that are equivalent to such a decomposition of the utility function clarify what is assumed by the model. In this case, we couple standard requirements of weak order, continuity and nontriviality with an axiom called cross-consistency, directing attention to the latter as capturing our departure from standard models. In addition, the axiomatization may in turn facilitate further analysis and testing of the model, as it may be easier to design and conduct empirical or experimental exercises that focus on this axiom than taking on the entire memorable-consumption package at once. Finally, we believe that, when interpreting x_2 as the memory-generating good, the axioms we impose are quite plausible, supporting our belief that the functional form is neither too peculiar nor ad hoc.

Our result applies to more general set-ups, and axiomatizes *quasi-separable* utility functions, defined as utility functions that can be written as the sum of two functions, each of which depends on a proper subset of the variables, where these subsets are not disjoint.[12]

4.1. The setting

Let X, Y, Z be convex subsets of Euclidean spaces. Denote their product by

$$A = X \times Y \times Z$$

and endow it with the product topology. We are interested in binary relations $\succsim \subset A \times A$ that can be represented by maximization of a function

$$f(x, y, z)$$

that can be written as

$$f(x, y, z) = u(x, y) + v(y, z)$$

where

$$u : X \times Y \to \mathbb{R}$$

and

$$v : Y \times Z \to \mathbb{R}$$

are continuous, non-constant functions.

In the memorable-good application, X is the bundle of ordinary goods; Y is the bundle of memorable goods consumed at present; and Z consists of bundles of memorable goods consumed in the past (or the corresponding levels of Υ_t and Λ_t defined by them). Clearly, the same structure can be used for other applications as well.

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[12] See Fishburn (1970, Theorem 11.3) for decompositions in a similar spirit.

4.2. The axioms

For a binary relation $\succsim \subset A \times A$ (with $A = X \times Y \times Z$) we state the following axioms:

A1. Weak order: $\succsim$ is complete and transitive.

A2. Continuity: For every $a \in A$, the sets $\{b \in A \mid b \succ a\}$, $\{b \in A \mid a \succ b\}$ are open.

A3. Cross-Consistency: For every $y_0, y_1 \in Y$, every $x_1, x_2, x_3, x_4 \in X$, and every $z_1, z_2, z_3, z_4 \in Z$, if

$$(x_1, y_0, z_1) \succsim (x_3, y_1, z_3)$$
$$(x_2, y_0, z_1) \precsim (x_3, y_1, z_4)$$
$$(x_1, y_0, z_2) \precsim (x_4, y_1, z_3)$$

then

$$(x_2, y_0, z_2) \precsim (x_4, y_1, z_4).$$

A4. Essentiality: For every $y \in Y$, there exist $x_1, x_2 \in X$ and $z \in Z$ such that $(x_1, y, z) \succ (x_2, y, z)$ and there exist $x \in X$ and $z_1, z_2 \in Z$ such that $(x, y, z_1) \succ (x, y, z_2)$.

The first two axioms and the final axiom are obvious and intuitive. We thus assume that $\succsim$ is a continuous weak order—which is necessarily the case for any continuous representation—and that $\succsim$ satisfies a sensitivity assumption, so that, given any value of y, neither of the other variables will be immaterial. (We comment on the importance of this assumption in the course of the proof.)

To understand the meaning of Cross-Consistency, consider first the pair of preferences

$$(x_1, y_0, z_1) \succsim (x_3, y_1, z_3)$$
$$(x_2, y_0, z_1) \precsim (x_3, y_1, z_4).$$

For the sake of the argument, imagine that preferences are monotonic in all coordinates, that x_2 is better than x_1, and that z_4 is better than z_3. On the left side, x_1 was replaced by x_2. On the right side, z_3 was replaced by z_4. As a result, the right side, which used to be not as highly ranked as the left side, became at least as good as the (modified) left side. This means, intuitively, that the difference in utility between z_4 and z_3 (at the level y_1) is at least as high as the difference in utility between x_2 and x_1 (at the level y_0).

Next consider the pair

$$(x_1, y_0, z_1) \succsim (x_3, y_1, z_3)$$
$$(x_1, y_0, z_2) \precsim (x_4, y_1, z_3).$$

Again, to understand the intuition, assume that z_2 is better than z_1 and the same holds for x_4 and x_3. By similar reasoning, the difference in utility between x_4 and x_3 (at the level y_1) is at least as high as the difference in utility between z_2 and z_1 (at the level y_0).

Finally, consider the first preference,

$$(x_1, y_0, z_1) \succsim (x_3, y_1, z_3),$$

and the question of which way the fourth should go, or

$$(x_2, y_0, z_2) ??? (x_4, y_1, z_4).$$

Moving from the first to the second of these comparisons, we see two improvements on the left side: x_1 was replaced by x_2 *and* z_1 was replaced by z_2. However, there are also two improvements

on the right side: x_3 was replaced by x_4 and z_3 was replaced by z_4. The left side improvements occur at the level $y = y_0$ and those on the right side occur at the level $y = y_1$. But these are precisely the levels of y for which we have some information from the first two comparisons. And since we know that the z_3–z_4 improvement (at y_1) beats the x_1–x_2 improvement (at y_0) *and* that the x_3–x_4 improvement (at y_1) beats the z_1–z_2 improvement (at y_0), we expect that the addition of the (respective) former will beat the addition of the (respective) latter, that is, that $(x_2, y_0, z_2) \precsim (x_4, y_1, z_4)$.

We thus find Cross-Consistency a reasonably compelling property. Section 6.4 exploits this line of argument to achieve a rather straightforward demonstration that Cross-Consistency is necessary for our representation. The main point of Proposition 7 is that together the additional mild assumptions imposed by the other axioms presented above, Cross-Consistency is also sufficient for the representation.

Remark. One may consider a weaker version of Cross-Consistency that is restricted to a single y level, that is, a version that requires $y_0 = y_1$. As will be clear from the proof, such a version implies Debreu's (1960) "Double Cancellation" axiom and is the basic driving force behind additive separability at each level of y. It is not hard to see, however, that such a weaker version would not suffice for our purposes. For example, assume that x, y, z are positive real variables, and that $\succsim$ is defined by maximization of

$$f(x, y, z) = y \log (x + z).$$

Clearly, at each level of y preferences are defined by maximization of $\log (x + z)$ or, equivalently, of $x + z$ and are therefore separable. Yet, it is not hard to see that such preferences cannot be represented by $u(x, y) + v(y, z)$ over the entire space, as they will not satisfy the necessary condition of Cross-Consistency. □

4.3. A representation result

Section 6.4 proves:

Proposition 7. *The relation $\succsim \subset A \times A$ satisfies A1–A4 if and only if there are continuous functions*

$$u : X \times Y \to \mathbb{R}$$
$$v : Y \times Z \to \mathbb{R}$$

such that $\succsim$ is represented by

$$f(x, y, z) = u(x, y) + v(y, z)$$

and such that, for each $y \in Y$, neither $u(\cdot, y)$ nor $v(y, \cdot)$ is a constant. Furthermore, in this case u and v are unique in the following sense: u' and v' also satisfy the representation above iff there are $\alpha > 0$, a continuous function $\beta : Y \to \mathbb{R}$, and $\gamma \in \mathbb{R}$ such that

$$u'(x, y) = \alpha u(x, y) + \beta(y)$$
$$v'(y, z) = \alpha v(y, z) - \beta(y) + \gamma.$$

When interpreting this representation result in the context of memorable consumption, one may take the elements of Z to be vectors of past consumption, so that preferences are represented by the function

$$u(x_{1t}, x_{2t}) + \tilde{v}_t(x_{20}, \ldots, x_{2t}).$$

The specific assumptions we impose on $\tilde{v}_t$, namely, the way that it depends on $x_{20}, \ldots, x_{2(t-1)}$ only through Υ_t, Λ_t, are then functional form assumptions. We could seek axiomatic foundations for this functional form, but do not expect such an axiomatization to add significantly to our understanding of the model.

5. Discussion

5.1. Related models

We discuss here the more prominent of the many other ways one can imagine current utility depending on past consumption, or that lumpiness in consumption might arise. Our basic model, given by (4), is general enough to capture these as special cases. It is the more particular assumptions made in Section 2.2 that make ours a model of memorable consumption, and that distinguish it from more familiar models.

5.1.1. Memorable goods vs. durable goods

Memorable consumption goods generate a subsequent flow of utility, just as do durable goods. Have we simply expanded the list of durable goods? There are some important differences. To compare, let us first construct a simple model of consumption with a durable good that sticks as closely as possible to our model of memorable goods. We interpret good 1 as a perishable consumption good and good 2 as a durable good. Utility is derived in each period from the consumption of good 1 in that period and from the consumption of a stock of the durable good. The consumer's objective is to maximize

$$\sum_{t=0}^{\infty} \delta^t u(x_{1t}, K_{2t}),$$

subject to the budget constraint

$$Y_{t+1} = [Y_t - x_{1t} - x_{2t}]\frac{1}{\delta} \geq 0,$$

where x_{2t} is the expenditure on the durable good in period t and K_{2t} is the period t stock of the durable good, with

$$K_{2t} = \lambda K_{2t-1} + (1 - \lambda)x_{2t-1}.$$

In this case, the optimal consumption plan will converge to a steady state in which $x_{1t} = x_1^*$ and $x_{2t} = x_2^*$ for some (x_1^*, x_2^*). Consumption is perfectly smoothed, unlike the case of memorable goods.

The previous example may make the durable good too pliable, essentially stripping the model of the lumpiness characteristically associated with durable goods. To construct a simple alternative, suppose that in any period the consumer can consume either one unit or no unit of the durable good. Once purchased, the durable good remains intact for a randomly drawn number of

periods before disintegrating completely. The consumer then faces the choice of either buying a new (randomly lived) durable good or going without the durable good.

To model this situation, we let $\overline{u}(x_1)$ be the utility derived from consuming x_1 of the ordinary consumption good as well as the durable good. Let $\underline{u}(x_1)$ be the utility gained from consuming x_1 of the ordinary consumption good and none of the durable good. Let $\overline{V}(Y)$ be the optimal value of the future consumption stream, given that the consumer currently owns a unit of the durable good and has a lifetime income whose present value is Y, and let $\underline{V}(Y)$ be the corresponding value when no good is owned. Let p be the price of the durable good. Then we have

$$\overline{V}(Y_t) = \max_{x_{1t}} \overline{u}(x_{1t}) + \lambda \overline{V}\left(\frac{Y_t - x_{1t}}{\delta}\right) + (1-\lambda)\underline{V}\left(\frac{Y_t - x_{1t}}{\delta}\right)$$

$$\underline{V}(Y) = \max_{x_{1t}, \mathbf{1}_t} \mathbf{1}_t\left[\overline{u}(x_{1t}) + \lambda \overline{V}\left(\frac{Y_t - x_{1t} - p}{\delta}\right) + (1-\lambda)\underline{V}\left(\frac{Y_t - x_{1t} - p}{\delta}\right)\right]$$
$$+ (1 - \mathbf{1}_t)\left[\underline{u}(x_{1t}) + \underline{V}\left(\frac{Y_t - x_{1t}}{\delta}\right)\right],$$

where $\mathbf{1}_t$ is an indicator for purchasing the durable good in period t. It is then straightforward to find conditions under which the durable good is initially purchased, and is purchased after each subsequent period in which the previous durable disintegrates, until an unlucky stream of disintegrating durables pushes income so low that the durable good is never subsequently purchased.

Behind this result lies the fact that an expenditure on a durable good generates a stream of benefits that is independent of past expenditures. The flow of benefits from the purchase of a sixty-inch flat screen television is the same whether this is the first television one ever owned or whether it is a replacement of the previous sixty-inch flat screen television that failed last week. In the absence of such a television, listening to the radio does not increase the subsequent utility generated by the television. As a result, a consumer whose income is sufficiently large smooths consumption by invariably replacing a defunct durable, while a consumer whose income is sufficiently small also smooths consumption, but this time by never purchasing the durable. In contrast, an expenditure on the memorable good generates future utility flows only if the expenditure is sufficiently above the customary level. A consumer who plans to spend a large amount on the memorable good next period might be better off if her consumption this period decreased, something not possible with durable goods. This gives rise to incentives to bunch consumption that do not arise with the durable good, and expenditure patterns that do not arise in the case of the durable. A consumer often delays her next consumption of a memorable good, but will never forego such consumption forever, while both of these properties are reversed in the case of a durable good. A spike in consumption with durable goods stems from a technological constraint while a spike in consumption of a memorable good is a choice which typically varies across consumers.

5.1.2. Indivisibilities

One might be concerned that the patterns we associate with memorable consumption simply reflect indivisibilities in consumption goods. There may be a minimum amount one has to spend on a wedding, or carnival costume, or vacation.

The interpretation of this possibility depends importantly on the source of the indivisibilities. One possibility is that these indivisibilities arise out of technological factors. One cannot splurge on half of a bungee jump or half of a skydive, and vacations may entail fixed costs such as

plane tickets. We would expect these costs to be reflected in consumption—people with higher incomes would purchase more expensive indivisible goods. A person near the median income may occasionally treat herself to dinner and a show, while a person in the top tenth of a percent may be one of the first space tourists. In either case, we would see some spikes in consumption. However, if only indivisibilities and not memorable consumption are at work, we would expect the person to otherwise perfectly smooth consumption, and (unlike the case of memorable consumption) would see no interaction between the expenditures on the indivisible good and other expenditures. In addition, if the indivisible goods are memorable, we would expect the spikes in consumption to be exacerbated. Consider an agent who has just returned from a week-long trip to Greece. Even if there are no monetary or time constraints, she might choose not to immediately take another week-long trip, but instead defer the trip until the utility flows she has just generated have faded. A fixed cost alone may thus give rise to a spike in expenditure on a memorable good, but we expect these spikes to be more pronounced when the goods are memorable.

Alternatively, suppose that the minimum expenditure required on an indivisible good arises out of an interaction with the consumer's income rather than technological considerations. It may be that one can spend any amount on a wedding, ranging from a virtually free ceremony at a Justice of the Peace to inviting your closest 500 friends to the party of a lifetime. Similarly, one can spend any amount on a vacation, from taking an afternoon off to walk on the beach to a round-the-world-cruise. However, it may also be that the resulting consumption contributes to utility only if it is sufficiently lavish compared to one's usual consumption. We can capture this in a model where the intertemporal objective of the consumer is given by:

$$\sum_{t=0}^{\infty} \delta^t \left[u(x_{1t}, x_{2t}) + v(x_{2t}, \Lambda_t) \right].$$

This differs from our memorable consumption model only in that the function $v(x_{2t}, \Lambda_t)$ is now interpreted not as the discounted sum of future utility flows, but simply as a burst of current utility. This is a difference of interpretation only, with the models being otherwise identical.

The key to this interpretation is the link between customary consumption and the level of indivisibility faced by the consumer; this link takes center stage in our analysis. We regard the presence of such a link as taking us outside the realm of what is typically meant by an indivisibility. We also regard the memorable nature of the consumption goods involved as a particularly plausible source of such a link.

5.1.3. Addiction

The presence of the stock Λ_t in our memorable utility specification prompts a comparison to models of addiction, with Becker and Murphy (1988) being the obvious counterpart. In their model, the utility function in period t is given by

$$u(y_t, c_t, \Lambda_t),$$

where y_t is the consumption of a nonaddictive good, c_t is the consumption of an addictive good, and Λ_t is the stock of the addictive good. We can assume this stock evolves according to $\Lambda_{2t} = \lambda \Lambda_{2t-1} + (1 - \lambda)c_{t-1}$.

A first departure from our model of memorable utility is that the stock Λ_t is allowed to enter the utility function with either a positive or negative sign, so that an increased stock of consumption may decrease utility (perhaps with something like smoking) or increase utility (perhaps with something like exercise). In addition, the cross derivative $u_{c\Lambda}$ may be either positive or negative,

so that an increased stock may either enhance or attenuate the urge for current consumption. The primary difference is that the function u in the addiction model as assumed to be concave. This allows a straightforward optimization in each period, and contrasts with the nonconcavities that appear in our case. Once again the distinction is that there is no counterpart in the addiction model to the premium on lumpy consumption that appears in the case of memorable goods.

5.1.4. Habit formation

If we eliminate Υ from the model and write the second component of our utility function as

$$\tilde{v}_t(x_{20}, \ldots, x_{2t}) = \hat{v}(x_{2t}, \Lambda_t),$$

then the effect of past consumption on the immediate utility from expenditure on the memorable good x_{2t} depends only on Λ_t. There then exist simple specifications for the evolution of Λ for which the model becomes a model of habit formation. Once again, however, the distinction arises out of the specification of the function $\hat{v}$. In a typical habit formation model, this function is strictly increasing in x_{2t} and is concave. In the case of memorable goods, this function increases in x_{2t} only after x_{2t} hits an extraordinary level, precluding the concavity of $\hat{v}$. Again, the behavioral distinction is between that of a tendency to smooth consumption in the case of habit formation (albeit at a different level than would appear without habit formation) and to bunch consumption in the case of memorable consumption.

5.1.5. Anticipation

Once one has incorporated the idea that an agent can enjoy pleasant thoughts of consumption long after the physical act of consumption, it is natural to consider pleasant thoughts *before* consumption, that is "anticipation utility". Indeed, as Loewenstein (1987) has emphasized, there are many consumption experiences (among others, he offers the example of a chance to kiss a movie star of your choice) that one might deliberately postpone in order to savor the anticipation.

There is an important difference between anticipation utility and memorable utility. Anticipation gives rise to a game between the various selves that govern consumption at various times. If I enjoy the anticipation of a meal at a three star restaurant for a full month prior to eating there, why not postpone the meal for another month when the day comes, in order to enjoy yet more anticipation? Carrying this possibility to its logical conclusion, one might well never consume the meal. However, one would then presumably anticipate that the meal will not occur, undermining the intervening flows of anticipation utility.

Anticipation thus gives rise to problems of time inconsistency that do not arise in our model of memorable consumption. How long can consumption be postponed and still generate utility from anticipation? Perhaps forever in some cases, though such people are likely to be regarded as delusional. In general, we expect that utility can be generated by anticipating consumption only if there is a reasonable belief that the consumption will indeed occur. But when the time comes, what induces the current decision maker to go through with the consumption when many of its benefits have already been received? Perhaps because reneging will have an overwhelmingly negative impact on current utility? Perhaps because of the realization that if one reneges, no future utility form anticipation can be generated? These are interesting questions, as is the general question of anticipation, but the resulting time-consistency considerations make this a game rather than a decision problem, requiring a different analysis.

5.2. Applications of a memorable utility model

5.2.1. Permanent income hypothesis

The standard intertemporal consumption model suggests that optimal consumption should be smooth. In particular, expected but temporary jumps in income have little effect on permanent income, and so should have little effect on consumption. For example, expected tax refund receipts should lead to little immediate increase in consumption. In contrast, Souleles (1999) documents that there is excess sensitivity of consumption to such refunds.

Hai et al. (2014) demonstrate that if the model in Souleles is extended along the lines of our model, there is essentially no excess sensitivity. Much of what appears to be a puzzling current consumption binge in response to temporary income shocks can be interpreted as memorable utility, which in turn generates a relatively smooth intertemporal pattern of increases in utility.

5.2.2. Memorable consumption as a substitute for saving

It is a familiar lament in the popular press that Americans save too little for retirement (see, for example, the opening quotation in Scholz et al. (2006)).[13] Our analysis of memorable goods brings a new dimension to this discussion, and a new reason to suspect that reports of undersaving may be overstated. The flow of utility in retirement years includes the flow of utility from earlier memorable consumption. Decreasing consumption expenditures as one moves into and through retirement, often taken as a sign of undersaving, may simply reflect the optimal management of the consumption of memorable goods. This suggests a number of empirical projects. For example, with sufficient data, one could look for a correlation between early expenditure on memorable goods and decreases in expenditure later in life that our model would suggest.

5.2.3. Estimating discount factors

Taking account of memorable goods in a consumer's utility maximization problem changes how we think about saving and investment in general. Separate from the question of retirement savings, economists have puzzled over why the savings rate in the U.S. has dropped substantially over the past half century (e.g., Parker, 2000). The decrease in the savings rate is sometimes interpreted as an indication that consumers care less about the future than they once did. However, over the period in which savings rates decreased, expenditures on vacations increased, providing a hint of a link between the generation of the consumption of memorable goods and savings.

A consumer who shifts expenditure from nonmemorable goods to memorable goods is doing something akin to saving—making current choices that increase her future utility. Any estimate of intertemporal preferences from longitudinal consumption that ignores the memory component of nondurable consumption will result in an upward bias of the consumer's discount rate. There is evidence that recreation is a luxury good (though perhaps becoming less so; see Costa, 1997). If memorable goods more generally are luxury goods, then estimates of discount factors may become more problematic for higher incomes. It would be interesting to revisit estimates of discount factors with an empirical strategy incorporating memorable goods.

5.2.4. Memorable goods and risk aversion

Ignoring the memory component of consumption will complicate estimates of risk aversion as well as estimates of discount rates. We discussed above the Friedman–Savage anomaly of agents

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[13] Whether this is actually the case is less clear. Scholz et al. (2006), for example, argue that only about twenty percent of Americans are undersaving, and even then not by vast amounts.

who both gamble and insure. If one accepts our model of memorable utility, a nonconvexity arises naturally, and the insurance-and-gambling behavior is less surprising. Section 3.4 showed that a consumer who understands the (possibly large) memorable utility that accompanies a big increase in consumption may optimally reduce current consumption for some time so as to be able to afford the memorable event. For our infinitely lived consumer, this intertemporal substitution is a sufficiently powerful tool for managing memorable utility as to obviate the need to take on risk. However, a real-world finitely-lived consumer might realize that she will not live long enough to acquire the resources needed to generate a large burst of memorable utility. A fair (or even mildly unfair) lottery with a large upside possibility may be part of an optimal plan for such a consumer even if the consumer is otherwise risk averse.

5.3. Extensions

We touch here on one of the many possible extensions of our model. We have laid out the model in which higher-than-customary consumption leads to future utility flows that are added to the direct contemporaneous utility from consumption. This captures well the positive utilities that stem from high expenditures, but the basic ideas can be extended to cover negative utilities as well. Most of us have at some time stayed at hotels that are memorable, but not in a positive way. If we were advising a friend on choosing a hotel for his honeymoon we would suggest paying a premium to be sure that the hotel didn't fall below expectations, since if the experience is negative its unpleasant effects will linger long afterward.

Accommodating negatively memorable consumption would be straightforward. The function v aggregates the flow of utility stemming from memorable consumption that is then added on to the direct utility from consumption. One can allow the function to take on negative values for unpleasant consumption experiences, which then have an ongoing drag on future utility. As with the positive effects of memorable consumption that we have modeled, the natural way to proceed would be to say that if the expenditure on the memorable good falls sufficiently below the customary level, negative utility flows are generated. The consumer may then appear to be loss averse over sufficiently large losses.

6. Appendix: proofs

6.1. Proof of *Lemma 2*

We first note that in any equilibrium, we have $(x_{10}, x_{20}) \in [0, Y_0]^2$. This implies that there is a lower bound $\varepsilon > 0$ on $\frac{du(x_{10}, x_{20})}{dx_{10}}$, the marginal utility of good 1 in the first period.

[STEP 1] There is an $\hat{x} \geq 0$ such that for any consumption (x_1, x_2) with both $x_1 > \hat{x}$ and $x_2 > \hat{x}$, it must be that either $\frac{du(x_1, x_2)}{dx_1} < \varepsilon/3$ or $\frac{du(x_1, x_2)}{dx_2} < \varepsilon/3$. (If this is not the case, then we can find a sequence $(x_{1\tau}, x_{2\tau})$ contradicting Assumption 3.4.) In addition, there exists a value $\check{x}$ such that if $x_2 > \check{x}$, then $\frac{d\hat{v}(x_2, \Lambda)}{dx_2} < \frac{\varepsilon}{3}(1 - \delta)$ for any Λ (where we see later in the proof why the $(1 - \delta)$ is needed to cope with the durability of memorable consumption). This follows from Assumption 3.5.

Let $\tilde{x} = \max\{\hat{x}, \check{x}\}$.

[STEP 2] Now suppose we have a candidate equilibrium in which, for some t, we have consumption bundle (x_{1t}, x_{2t}) with both $x_{1t} > \tilde{x}$ and $x_{2t} > \tilde{x}$. We establish a contradiction by showing that there exists a superior consumption plan in which consumption is unchanged in all periods except 0 and t, and in which consumption of each good in periods 0 and t falls short of $\tilde{x}$.

Notice that, by [STEP 1], in period t, the derivative of u with respect to either good 1 or good 2 must fall short of $\varepsilon/3$.

[STEP 2.1] Suppose the derivative with respect to good 1 that has this property. Then we have

$$u_1(x_{10}, x_{20}) - u_1(x_{1t}, x_{2t}) > \frac{2}{3}\varepsilon. \tag{16}$$

We now consider an alternative strategy which duplicates the candidate equilibrium strategy, with the exception of x_{10} and x_{1t}. The latter two variables are allowed to vary (giving us the collection of alternative strategies as they do so) as long as they satisfy the budget constraint. The budget constraint in turn requires

$$x_{10} + \delta^t x_{1t} = k$$

for some constant k. This allows us to define an implicit function $x_{1t} = f(x_{10})$ whose derivative is given by $-\delta^{-t}$. This allows us to interpret the left side of (16) as the derivative of the discounted sum of utility with respect to x_{10}, under the constraint that $x_{1t} = f(x_{10})$, ensuring the budget constraint is satisfied. The fact that this derivative is positive ensures that we can increase utility by decreasing x_{1t} and increasing x_{10}.

Now suppose instead that the derivative of u with respect to good 2 falls short of $\varepsilon/3$. Then, from [STEP 1]

$$u_1(x_{10}, x_{20}) - \delta^t\left[u_2(x_{1t}, x_{2t}) + \sum_{\tau=t}^{\infty}\delta^{\tau-t}\hat{v}_1(x_{2t}, \Lambda_t)\right] =$$

$$u_1(x_{10}, x_{20}) - \delta^t\left[u_2(x_{1t}, x_{2t}) + \frac{1}{1-\delta}\hat{v}_1(x_{2t}, \Lambda_t)\right] > \frac{1}{3}\varepsilon.$$

This sum is an upper bound on the derivative of the discounted sum of utility with respect to x_{10}, under the constraint that $x_{2t} = f(x_{10})$. (In particular, it captures the effect of varying x_{10} on first-period utility, and then the effect of varying x_{2t} on utility in period t and every subsequent period. It is only an upper bound, because the derivative is reduced by the effect of x_{2t} on values of Λ_τ for values $\tau > t$, which we have not incorporated in our calculation.) The fact that this derivative is positive again ensures that we can increase utility by decreasing x_{2t} and increasing x_{10}.

This allows us to construct a sequence of improvements that continues until at least one of x_{1t} of x_{2t} is no larger than $\tilde{x}$ for all t. We can apply this argument to any period t, and hence an optimal consumption plan cannot feature any period t in which both x_{1t} and x_{2t} exceed $\tilde{x}$.

[STEP 3] It follows from Assumption 3.3, Assumption 3.2, the continuous differentiability of u, and Dini's theorem that there exists x' such that if $x_1 > x'$ and $x_2 \in [0, \tilde{x}]$, then $\frac{du(x_1,x_2)}{dx_1} < \varepsilon/3$. We can then repeat the argument of [STEP 2] to obtain a contradiction, ensuring that an optimal consumption plan can exhibit no period t in which $x_1 > x'$ and $x_2 \in [0, \tilde{x}]$.

[STEP 4] It follows from Assumption 3.3, Assumption 3.2, the continuous differentiability of u, and Dini's theorem that there exists x'' such that if $x_2 > x''$ and $x_1 \in [0, \tilde{x}]$, then $\frac{du(x_1,x_2)}{dx_2} < \frac{\varepsilon}{3}(1-\delta)$. We can assume x'' is at least as large as $\tilde{x}$, and then repeat the argument of [STEP 2] to obtain a contradiction, ensuring that an optimal consumption plan can exhibit no period t in which $x_2 > x''$ and $x_1 \in [0, \tilde{x}]$

We then complete the argument by taking $\overline{x} = \max\{\tilde{x}, x', x''\}$.

6.2. Proof of Proposition 4

Let $\{\hat{x}_{1t}, \hat{x}_{2t}\}_{t=0}^{\infty}$ be the optimal consumption plan. By assumption this plan involves memorable consumption in some period, though we cannot be sure of which period. An alternative consumption plan is to never undertake memorable consumption, in which case the optimal plan would be $\{(x_1^*, x_2^*), (x_1^*, x_2^*), \ldots\}$. We assume that it is strictly optimal to undertake memorable consumption at least once, giving

$$U\left(\{\hat{x}_{1t}, \hat{x}_{2t}\}_{t=0}^{\infty}\right) > U\left(((x_1^*, x_2^*), (x_1^*, x_2^*), \ldots)\right). \tag{17}$$

The optimal consumption plan induces a sequence of stocks $\{\hat{\Lambda}_t\}_{t=0}^{\infty}$ and wealths $\{\hat{Y}_t\}_{t=0}^{\infty}$.

Suppose the optimal plan involved memorable consumption only finitely many times. Then there is a period T such that the consumer enters period $T+1$ with wealth Y_{T+1}, and thereafter consumes

$$\left(x_1^* \frac{Y_T}{Y_0}, x_2^* \frac{Y_T}{Y_0}\right)$$

in each period. As a result, the stock Λ_t becomes arbitrarily close to $x_2^* \frac{Y_T}{Y_0}$. It then suffices to show that if a period t' arrives in which $\Lambda_{t'} = x_2^* \frac{Y_T}{Y_0}$, then the consumer will indulge in memorable consumption in some subsequent period.[14] To see that this is the case, notice that once such a t' has arrived, the continuation sequences

$$\left\{\hat{x}_{1t} \frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}, \left\{\hat{x}_{2t} \frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}, \left\{\hat{\Lambda}_t \frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}, \left\{\hat{Y}_t \frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}$$

are feasible and, by definition, involves memorable consumption. We then need only argue that

$$U\left(\left\{\hat{x}_{1t} \frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}, \left\{\hat{x}_{2t} \frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}\right) > U\left(\left(\hat{x}_1^* \frac{Y_T}{Y_0}, \hat{x}_2^* \frac{Y_T}{Y_0}\right), \left(\hat{x}_1^* \frac{Y_T}{Y_0}, \hat{x}_2^* \frac{Y_T}{Y_0}\right), \ldots\right).$$

But this follows from (17) and the homogeneity invoked in Assumption 4.

6.3. Proof of Proposition 6

Fix an optimal consumption plan, and let there be no memorable consumption in periods t and $t+1$. Let the periods after $t+1$ in which memorable consumption occurs be $\{\tau_k\}_{k=0}^{\infty}$. Then the derivative of the period-t continuation discounted sum of utility with respect to x_{2t} is given by

$$u_2(x_{1t}, x_{2t}) + \delta^{\tau_0 - t} \sum_{k=0}^{\infty} \delta^{\tau_k - \tau_0} v_2(x_{2\tau_k}, \Lambda_{\tau_k}) \frac{d\Lambda_{\tau_k}}{dx_{2t}},$$

while the corresponding derivative with respect to x_{2t+1} is given by

$$u_2(x_{1t+1}, x_{2t+1}) + \delta^{\tau_0 - (t+1)} \sum_{k=0}^{\infty} \delta^{\tau_k - \tau_0} v_2(x_{2\tau_k}, \Lambda_{\tau_k}) \frac{d\Lambda_{\tau_k}}{dx_{2t+1}}.$$

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[14] We can only be assured that $\Lambda_{t'}$ can be made arbitrarily close to $x_2^* \frac{Y_T}{Y_0}$, not precisely equal to it, but the fact that our original preference is strict allows us to look at the case where they are equal.

In each case the first term captures the immediate effect of consuming good 2 on current consumption, and the summation captures the effect on the future customary levels of consumption, which come into play each time memorable consumption occurs. We can rewrite these derivatives as

$$u_2(x_{1t}, x_{2t}) + \delta^{\tau_0 - t} \sum_{k=0}^{\infty} \delta^{\tau_k - \tau_0} v_2(x_{2\tau_k}, \Lambda_{\tau_k})(1 - \lambda)\lambda^{\tau_k - t - 1},$$

and

$$u_2(x_{1t+1}, x_{2t+1}) + \delta^{\tau_0 - (t+1)} \sum_{k=0}^{\infty} \delta^{\tau_k - \tau_0} v_2(x_{2\tau_k}, \Lambda_{\tau_k})(1 - \lambda)\lambda^{\tau_k - (t+1) - 1}.$$

The necessary conditions for utility maximization are that these two derivatives be equal. Noting that $v_2 < 0$, this gives

$$u_2(x_{1t}, x_{2t}) < u_2(x_{1t+1}, x_{2t+1}).$$

6.4. Proof of Proposition 7

6.4.1. Necessity

Assume that $u(x, y) + v(y, z)$ represents $\succsim$. A1 and A2 follow immediately. As for A3, assume that

$$(x_1, y_0, z_1) \succsim (x_3, y_1, z_3)$$
$$(x_2, y_0, z_1) \precsim (x_3, y_1, z_4)$$
$$(x_1, y_0, z_2) \precsim (x_4, y_1, z_3).$$

The first preference statement implies

$$u(x_1, y_0) + v(y_0, z_1) \geq u(x_3, y_1) + v(y_1, z_3),$$

or, equivalently,

$$-u(x_1, y_0) - v(y_0, z_1) \leq -u(x_3, y_1) - v(y_1, z_3),$$

while the other two yield

$$u(x_2, y_0) + v(y_0, z_1) \leq u(x_3, y_1) + v(y_1, z_4)$$
$$u(x_1, y_0) + v(y_0, z_2) \leq u(x_4, y_1) + v(y_1, z_3).$$

Summing up the last three inequalities we obtain

$$u(x_2, y_0) + v(y_0, z_2) \leq u(x_4, y_1) + v(y_1, z_4)$$

which implies

$$(x_2, y_0, z_2) \precsim (x_4, y_1, z_4).$$

Finally, observe that A4 holds provided that $u(\cdot, y)$ and $v(y, \cdot)$ are not constant for any y.

6.4.2. Sufficiency – part I: construction

In this subsection we construct u, v such that $u(x, y) + v(y, z)$ represents preferences. We will fix $x_0 \in X$ and construct these functions so that

$$u(x_0, y) = 0 \qquad \forall y \in Y.$$

This will prove useful in showing that the functions so constructed are continuous (in Part II), as well as in proving the uniqueness result.

Step 0: Preliminaries: Cross Consistency has a natural counterpart, with the direction of all preference signs reversed:

Reverse Cross Consistency: For every $y_0, y_1 \in Y$, every $x_1, x_2, x_3, x_4 \in X$, and every $z_1, z_2, z_3, z_4 \in Z$, if

$(x_1, y_0, z_1) \precsim (x_3, y_1, z_3)$

$(x_2, y_0, z_1) \succsim (x_3, y_1, z_4)$

$(x_1, y_0, z_2) \succsim (x_4, y_1, z_3)$

then

$(x_2, y_0, z_2) \succsim (x_4, y_1, z_4).$

Note that the two conditions are equivalent. (To see this, it suffices to exchange the notation between $y_0 \leftrightarrow y_1, x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4, z_1 \leftrightarrow z_3, z_2 \leftrightarrow z_4$.)

Similarly, one can have the indifference version of the axiom:

Indifference Cross Consistency: For every $y_0, y_1 \in Y$, every $x_1, x_2, x_3, x_4 \in X$, and every $z_1, z_2, z_3, z_4 \in Z$, if

$(x_1, y_0, z_1) \sim (x_3, y_1, z_3)$

$(x_2, y_0, z_1) \sim (x_3, y_1, z_4)$

$(x_1, y_0, z_2) \sim (x_4, y_1, z_3)$

then

$(x_2, y_0, z_2) \sim (x_4, y_1, z_4).$

This version follows from the conjunction of Cross Consistency and Reverse Cross Consistency, and hence from each of these alone.

Step 1: Additive representation for any fixed y: For $y \in Y$, define

$$A_y = \{(x, y, z) \in A \mid x \in X, z \in Z\}.$$

Restricting attention to A_y, for each $y \in Y$, we note that $\succsim$ is a continuous weak order (basically, on $X \times Z$). For a relation $\succsim$ on $X \times Z$ we will be interested in following condition[15]:

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[15] See also Blaschke (1927) and Thomsen (1927) for the related "hexagon" condition.

Double Cancellation: For every $f, g, h \in X$ and every $p, r, q \in Z$, if

$$(f, p) \succsim (g, q)$$

and

$$(h, p) \precsim (g, r)$$

then

$$(h, q) \precsim (f, r).$$

In particular, the following lemma states that $\succsim$ on A_y satisfies Double Cancellation.

Lemma 8. *For each $y \in Y$, every $f, g, h \in X$ and every $p, r, q \in Z$, if*

$$(f, y, p) \succsim (g, y, q)$$

and

$$(h, y, p) \precsim (g, y, r)$$

then

$$(h, y, q) \precsim (f, y, r).$$

Proof. Given $f, g, h \in X$ and $p, r, q \in Z$ that satisfy $(f, y, p) \succsim (g, y, q)$, and $(h, y, p) \precsim (g, y, r)$, define (i)

$$y_0 = y_1 = y$$

(ii)

$$x_1 = x_4 = f$$
$$x_2 = h \quad x_3 = g$$

and (iii)

$$z_1 = p$$
$$z_2 = z_3 = q$$
$$z_4 = r.$$

Observe that

$$(x_1, y_0, z_1) = (f, y, p) \succsim (g, y, q) = (x_3, y_1, z_3)$$
$$(x_2, y_0, z_1) = (h, y, p) \precsim (g, y, r) = (x_3, y_1, z_4)$$

and clearly also

$$(x_1, y_0, z_2) = (f, y, q) \sim (f, y, q) = (x_4, y_1, z_3)$$

hence Cross Consistency can be invoked to conclude that

$$(x_2, y_0, z_2) \precsim (x_4, y_1, z_4)$$

which means

$$(x_2, y_0, z_2) = (h, y, q) \precsim (f, y, r) = (x_4, y_1, z_4). \qquad \square$$

Thus, Double Cancellation on each A_y follows from Cross Consistency. It follows from Debreu (1960) that $\succsim$ has an additively separable continuous representation: there are continuous $u_y : X \to \mathbb{R}$ and $v_y : Z \to \mathbb{R}$ such that, for every $x, x' \in X$ and every $z, z' \in Z$,

$$(x, y, z) \succsim \left(x', y, z'\right)$$

iff

$$u_y (x) + v_y (z) \geq u_y \left(x'\right) + v_y \left(z'\right).$$

Further, thanks to A4, these u_y, v_y are unique up to multiplication by a positive constant and the addition of a constant.[16] In other words, if $u'_y : X \to \mathbb{R}$ and $v'_y : Z \to \mathbb{R}$ also represent $\succsim$ on A_y as above, there must be $\alpha_y > 0$ and $\beta_{yu}, \beta_{yv} \in \mathbb{R}$ such that

$$u'_y (x) = \alpha_y u_y (x) + \beta_{yu}$$

and

$$v'_y (z) = \alpha_y v_y (z) + \beta_{yv}.$$

Finally, by setting $\beta_{yu} = -\alpha_y u_y (x_0)$ we may assume without loss of generality that $u_y (x_0) = 0$.

Step 2: Additive representation on the entire space: For each y, choose two continuous functions $u_y : X \to \mathbb{R}$ and $v_y : Z \to \mathbb{R}$ that represent $\succsim$ on A_y as above, with $u_y (x_0) = 0$. Recall that these functions are unique up to multiplication (of both) by (the same) $\alpha_y > 0$ and an addition of a constant to v_y. We now wish to show that we may take such an affine transformation of u_y, v_y for each $y \in Y$ so that the resulting functions rank alternatives as does $\succsim$ also across different values of y (that is, for any two alternatives $a \in A_y, b \in A_{y'}$ even if $y \neq y'$) and that these functions are continuous (over all of A).

For a generic $a \in A$, let $a_X \in X$, $a_Y \in Y$, and $a_Z \in Z$ be its components, so that $(a_X, a_Y, a_Z) = a$.

To visualize the construction, consider, for each $y \in Y$, the image of the functions u_y, v_y. Define

$$I (y) = \left\{ \left(u_y (x), v_y (z)\right) | x \in X, z \in Z \right\} \subset \mathbb{R}^2.$$

Note that, because u_y, v_y are continuous functions on convex subsets of Euclidean spaces, the images of these functions are convex. That is

$$I_u (y) \equiv \left\{ u_y (x) | x \in X \right\} \subset \mathbb{R}$$
$$I_v (y) \equiv \left\{ v_y (z) | z \in Z \right\} \subset \mathbb{R}$$

are (potentially infinite) intervals in $\mathbb{R}$, and $I (y) = I_u (y) \times I_v (y)$.

We may imagine indifference curves in $I (y)$, which are downward sloping straight lines with slope -1.

We define $y, y' \in Y$ to be *close* if there exist $x, x' \in X$ and $z, z' \in Z$ such that $(x, y, z) \sim \left(x', y', z'\right)$ while $\left(u_y (x), v_y (z)\right)$ is in the interior of $I (y)$ and $\left(u_{y'} \left(x'\right), v_{y'} \left(z'\right)\right)$ is in the interior of $I \left(y'\right)$.

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[16] If one of the two variables x, z does not affect preferences on A_y, its function is a constant—hence unique up to a cardinal transformation, but the representation of the other variable becomes only ordinal. Similarly, if A4 doesn't hold, one of the two functions might be a non-continuous monotone transformation of a continuous function.

Because $\succsim$ is known to be a continuous weak order on all of A, it can be represented by a continuous function $W : A \to \mathbb{R}$ (see Debreu, 1954). Restricting attention to A_y for any $y \in Y$, W is an increasing monotone transformation of $u_y + v_y$. The function W will allow us to simplify the notation in some of the following arguments, though it doesn't serve any particular role and, clearly, anything stated in the language of W can also be stated in the language of $\succsim$.

Observe that, for any $y \in Y$, $W(A_y)$ is a (potentially infinite) interval in $\mathbb{R}$ with a nonempty interior (due to A4). Moreover, if $y, y' \in Y$ are close, then $W(A_y) \cap W(A_{y'})$ is also a (potentially infinite) interval in $\mathbb{R}$ with a nonempty interior.

Lemma 9. *Assume that $y, y' \in Y$ are close. Then there are unique $\alpha_{y'} > 0$ and $\beta_{y'} \in \mathbb{R}$ such that, by defining*

$$u(x, y) = u_y(x); \quad u(x, y') = \alpha_{y'} u_{y'}(x)$$

and

$$v(y, z) = v_y(z); \quad v(y', z) = \alpha_{y'} v_{y'}(z) + \beta_{y'}$$

we obtain u and v such that $u(x, y) + v(y, z)$ represents $\succsim$ on $A_y \cup A_{y'}$.

Thus, the lemma states that we can fix the arbitrarily chosen u_y, v_y for one value, y, and choose an affine positive transformation of the functions for the other, y', and thus obtain a function that represents preferences not only within each subspace A_y, $A_{y'}$ but also across them.

Proof. Let $y, y' \in Y$ be close. Denote by I° the interior of $W(A_y) \cap W(A_{y'})$ which is known to be a nonempty (potentially infinite) interval in $\mathbb{R}$. Clearly, for any $\alpha_{y'} > 0$ and $\beta_{y'}^u, \beta_{y'}^v \in \mathbb{R}$

$$u(x, y) = u_y(x); \quad u(x, y') = \alpha_{y'} u_{y'}(x) + \beta_{y'}^u$$

and

$$v(y, z) = v_y(z); \quad v(y', z) = \alpha_{y'} v_{y'}(z) + \beta_{y'}^v$$

would result in a function $u(x, y) + v(y, z)$ that represents $\succsim$ on A_y as well as on $A_{y'}$. We need to make sure that such a function correctly represents $\succsim$ between $a \in A_y$ and $b \in A_{y'}$. To this end, we first focus on the W-inverse images of I°, that is on

$$\hat{A}_y \equiv \left\{ a \in A_y \mid W(a) \in I^\circ \right\}$$
$$\hat{A}_{y'} \equiv \left\{ b \in A_{y'} \mid W(b) \in I^\circ \right\}.$$

Consider a value $\xi = u_{y'}(x') + v_{y'}(z') \in \mathbb{R}$ for some $(x', y', z') \in \hat{A}_{y'}$. We claim that there exists a unique $\eta \in \mathbb{R}$ such that, for any $(x, y, z) \in \hat{A}_y$, $(x, y, z) \sim (x', y', z')$ if and only if $u_y(x) + v_y(z) = \eta$. Indeed, this follows from the fact that $u_y(x) + v_y(z)$ represents $\succsim$ on A_y, $u_{y'}(x') + v_{y'}(z')$—on $A_{y'}$, and from transitivity. Hence there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $(x, y, z) \sim (x', y', z')$ if and only if

$$u_y(x) + v_y(z) = g\left(u_{y'}(x') + v_{y'}(z')\right).$$

Further, by transitivity, g is increasing (and strictly increasing in the relevant domain) and

$$(x, y, z) \succsim (x', y', z')$$

iff

$$u_y(x) + v_y(z) \geq g\left(u_{y'}(x') + v_{y'}(z')\right).$$

We wish to show that g is affine on the relevant domain, that is on $u_{y'}(x') + v_{y'}(z')$ where $W\left((x', y', z')\right) \in I^\circ$. To this end, consider three equally-spaced points in its domain,

$$\xi, \xi + \delta, \xi + 2\delta.$$

We wish to show that the values of g for these points are also equally spaced, that is, that

$$g(\xi + 2\delta) - g(\xi + \delta) = g(\xi + \delta) - g(\xi).$$

Choose $x_1, x_2 \in X$, $z_1, z_2 \in Z$ such that

$$u_{y'}(x_1) + v_{y'}(z_1) = \xi$$
$$u_{y'}(x_2) + v_{y'}(z_1) = \xi + \delta$$
$$u_{y'}(x_1) + v_{y'}(z_2) = \xi + \delta.$$

Note that such a selection is possible since there are points in $A_{y'}$ with $u_{y'}(x) + v_{y'}(z) = \xi + 2\delta$. (Note that the selection of x_1, x_2, z_1, z_2 should be done simultaneously: there may be points x_1, z_1 close to the boundary of $A_{y'}$ for which such an x_2 or such a z_2 will not exist.)

Similarly, denote

$$\eta = g(\xi)$$
$$\varepsilon = g(\xi + \delta) - g(\xi) > 0$$

and choose $x_3, x_4 \in X$, $z_3, z_4 \in Z$ such that

$$u_y(x_3) + v_y(z_3) = \eta$$
$$u_y(x_4) + v_y(z_3) = \eta + \varepsilon$$
$$u_y(x_3) + v_y(z_4) = \eta + \varepsilon.$$

Clearly, we have

$$(x_1, y', z_1) \sim (x_3, y, z_3)$$
$$(x_2, y', z_1) \sim (x_3, y, z_4)$$
$$(x_1, y', z_2) \sim (x_4, y, z_3).$$

By Indifference Cross Consistency, we also have

$$(x_2, y', z_2) \sim (x_4, y, z_4)$$

which implies

$$g(\xi + 2\delta) = \eta + 2\varepsilon.$$

Thus g is affine on the relevant domain, and there are $\alpha_{y'} > 0$ and $\beta_{y'} \in \mathbb{R}$ such that

$$u_y(x) + v_y(z) = g\left(u_{y'}(x') + v_{y'}(z')\right)$$
$$= \alpha_{y'} u_{y'}(x') + \alpha_{y'} v_{y'}(z') + \beta_{y'}$$

whenever $(x, y, z) \sim (x', y', z')$.

Observe that in defining u and v we have some freedom in deciding how to split $\beta_{y'}$ between them. In fact, for any $\beta_{y'}^u, \beta_{y'}^v \in \mathbb{R}$ such that $\beta_{y'}^u + \beta_{y'}^v = \beta_{y'}$ we can define

$$u(x, y) = u_y(x); \quad u(x, y') = \alpha_{y'} u_{y'}(x) + \beta_{y'}^u$$

and

$$v(y, z) = v_y(z); \quad v(y', z) = \alpha_{y'} v_{y'}(z) + \beta_{y'}^v$$

and observe that $u(x, y) + v(y, z)$ represents $\succsim$ for any $a \in \hat{A}_y$ and $b \in \hat{A}_{y'}$. However, to stick to the normalization by which $u(x_0, \cdot) = 0$, we choose $\beta_{y'}^u = 0$ (recall that $u_y(x_0) = u_{y'}(x_0) = 0$) and $\beta_{y'}^v = \beta_{y'}$.

Next consider $a \in A_y \backslash \hat{A}_y$. If there exists $b \in A_{y'}$ such that $a \sim b$ (which might be possible if a and or b are $\succsim$-maximal or $\succsim$-minimal in their sub-spaces, A_y or $A_{y'}$, respectively), the proof continues as above, via transitivity. We are therefore left with the case that $a \succ A_{y'}$ or $a \prec A_{y'}$ (using the obvious notation for a relation between an element and every element of a set). But in this case one can choose $c \in \hat{A}_y$ and complete the proof by transitivity. (For example, for $b \in \hat{A}_{y'}$ one chooses $c \sim b$ and argues that $a \succ c \sim b$ occurs when $u(x, y) + v(y, z)$ obtains a higher value for a than both c and b; otherwise we may have $[a \succ A_{y'}$ and $b \precsim A_y]$ or $[a \prec A_{y'}$ and $b \succsim Ay]$ etc.) $\quad \square$

For $y \in Y$, let

$$C(y) = \left\{ y' \in Y \mid y \text{ and } y' \text{ are close} \right\}.$$

Lemma 10. *For every $y \in Y$ there exist*

$$u : X \times C(y) \to \mathbb{R}$$
$$v : C(y) \times Z \to \mathbb{R}$$

such that $u(x, y) + v(y, z)$ represents $\succsim$ on $\bigcup_{y' \in C(y)} A_{y'}$.

This lemma states that we can have the desired representation not only for every pair of subspaces A_y, $A_{y'}$ where y' is close to y, but also for all of these simultaneously (holding y fixed).

Proof. Let there be given $y \in Y$. For every $y' \in C(y)$ define $u(x, y') + v(y', z)$ as in the Lemma 9.

Consider $a, b \in \bigcup_{y' \in C(y)} A_{y'}$. If $a, b \in A_{y'}$ for some y' the proof is complete. This is also the case if one of them is in A_y. We are therefore left with the case that

$$a \in A_{y'} \backslash A_y \quad b \in A_{y''} \backslash A_y.$$

In this case we know that both a and b are either "above" all of A_y or "below" it. (That is, $a \succ A_y$ or $a \prec A_y$ and the same is true of b.) In case $a \succ A_y \succ b$ or $b \succ A_y \succ a$, transitivity completes the proof. Hence, we are interested in the case $a, b \succ A_y$ or, symmetrically, $a, b \prec A_y$. Without loss of generality assume that $a, b \succ A_y$. Since y' and y'' are both close to y and they both contain alternatives that are better than A_y, they have to be close to each other. In fact, there has to be a nonempty interior of

$$W(A_y) \cap W(A_{y'}) \cap W(A_{y''}).$$

Consider two real numbers in this interior, $\alpha < \beta$, and six alternatives c, c', d, d', e, e' such that $c, c' \in A_y$, $d, d' \in A_{y'}$, $e, e' \in A_{y''}$ and $W(c) = W(d) = W(e) = \alpha$ and $W(c') = W(d') = W(e') = \beta$. By Lemma 9, $u(\cdot, y')$ and $v(y', \cdot)$ are affine transformations of $u_{y'}(\cdot)$ and $v_{y'}(\cdot)$, respectively, and $u(\cdot, y'')$ and $v(y'', \cdot)$ are affine transformations of $u_{y''}(\cdot)$ and $v_{y''}(\cdot)$. However, the equalities above imply that, if we start with $u(\cdot, y')$ and $v(y', \cdot)$ and use Lemma 9 for y' and y'', we will end up with $u(\cdot, y'')$ and $v(y'', \cdot)$. Hence, $u(x, y) + v(y, z)$ represent preferences also on $A_{y'} \cup A_{y''}$ and correctly rank a and b. $\square$

Lemma 11. *Let there be given $y_1, y_2, \ldots, y_n \in Y$ such that y_i is close to y_{i+1} for $i = 1, \ldots, n-1$. Let*

$$C = \bigcup_{i \leq n} C(yi).$$

There exist

$$u : X \times C \to \mathbb{R}$$
$$v : C \times Z \to \mathbb{R}$$

such that $u(x, y) + v(y, z)$ represents $\succsim$ on $\bigcup_{y' \in C} A_{y'}$.

Proof. The proof is by induction on n, with the case $n = 1$ established by Lemma 10. Assume, then, that the claim is true for n and consider $n + 1$.

Fix u and v as provided for $y_1, y_2, \ldots, y_n$. Applying Lemma 10 to y_{n+1}, there are u' and v' defined on $C(y_{n+1})$ that represent $\succsim$ (by their sum) over all of $C(y_{n+1})$. The latter includes a nonempty W-value intersection with A_{y_n}, because y_{n+1} and y_n are close. This means that we can use an affine transformation of u' and v' that would be identical to u and v, respectively, over their intersection.

Clearly, the newly-extended u and v will represent preferences over $C(y_{n+1})$. To see that they do so for all of $\bigcup_{y' \in C} A_{y'}$ we use transitivity as before. $\square$

We are finally ready to complete the proof. We argue that there exists a double-sequence

$$\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$$

such that (i) y_i and y_{i+1} are close for $i \in \mathbb{Z}$; (ii) $Y = \bigcup_{i \in \mathbb{Z}} C(yi)$.

To see that this is the case, use the range of the function W as follows: first, consider a bounded interval $[-M, M]$ in the range of W. The interior of $W(A_y)$ for all $y \in Y$ is an open cover of $[-M, M]$, and thus has a finite subcover. From such a subcover one can choose a finite sequence $y_1, y_2, \ldots, y_n \in Y$ such that y_i is close to y_{i+1} for $i = 1, \ldots, n-1$ and that $[-M, M] \subset \bigcup_{i \leq n} W(Ay)$. Then, by induction on M one generated the sequence $\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$ such that (i) y_i and y_{i+1} are close for $i \in \mathbb{Z}$; (ii) $W(A) = \bigcup_{i \in \mathbb{Z}} W(Ay)$. Finally, one considers Lemma 11 and notes that in its inductive proof the functions u and v are defined as extensions of the same functions in previous steps. Repeating this argument for the doubly-infinite sequence completes the proof of existence of u and v. $\square$

6.4.3. Sufficiency – part II: continuity

We now turn to prove that the functions constructed above are continuous. Observe that for this to be true, one has to rely on the specific construction by which $u(x_0, y) = 0$, which guarantees that $u(x_0, \cdot)$ is continuous in y. Indeed, it is easy to see that by defining

$$u'(x, y) = \alpha u(x, y) + \beta(y)$$
$$v'(y, z) = \alpha v(y, z) - \beta(y)$$

for a discontinuous $\beta(\cdot)$, one can represent $\succsim$ by $u'(x, y) + v'(y, z)$ where neither u' nor v' are continuous, though their sum is.

Step 1: Continuity of $u + v$: It is convenient to rely on the continuous function W that represents $\succsim$. Since $u(x, y) + v(y, z)$ and W both represent $\succsim$, there exists a monotonically increasing $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$u(x, y) + v(y, z) = \phi(W(x, y, z))$$

for all $(x, y, z) \in X \times Y \times Z$. We claim that ϕ is continuous. Assume that it isn't, and that there exists $\xi_n \to \xi$ and $\varepsilon > 0$ such that

$$\phi(\xi_n) < \phi(\xi) - \varepsilon \tag{18}$$

or,

$$\phi(\xi_n) > \phi(\xi) + \varepsilon. \tag{19}$$

Consider first case (18). As ξ is in the domain of ϕ, it is in the range of W and thus $\xi \in W(A_y)$ for some y's. For each one of these, it has to be the case that $\xi = \min W(A_y)$. Otherwise, we can find $a_n, a \in A_y$ such that $a_n \to a$ but $\phi(W(a_n))$ fails to converge to $\phi(W(a))$, which is impossible because $u(x, y) + v(y, z)$ is continuous on each A_y separately.

As the range of W is connected, and it is the union of open intervals $\{W(A_y)\}_y$, it follows that

$$\xi = \min \cup_y W(A_y)$$

in which case (18) is impossible.

Similarly, in case (19) we conclude that $\xi = \max \cup_y W(A_y)$, and a contradiction results again. We therefore conclude that $u(x, y) + v(y, z) = \phi(W(x, y, z))$ is a continuous function on $A = X \times Y \times Z$.

Step 2: Continuity of u, v: We now wish to show that u is continuous on $X \times Y$. Clearly, this will mean that u is continuous on $X \times Y \times Z$ and therefore that so is

$$v(y, z) = \phi(W(x, y, z)) - u(x, y).$$

To this end we show that $u(x, y)$ is a continuous function of y, and that it is uniformly continuous with respect to x:

Lemma 12. *Let there be given $\tilde{x} \in X$ and $\tilde{y} \in Y$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ and $y \in Y$, if*

$$|x - \tilde{x}|, |y - \tilde{y}| < \delta$$

then

$$|u(x, y) - u(x, \tilde{y})| < \varepsilon.$$

Proof. Assume not. Then there are $\tilde{x} \in X$, $\tilde{y} \in Y$ and $\varepsilon > 0$ such that $\forall \delta > 0$ there are x, y that are δ-close to $\tilde{x}$, $\tilde{y}$, respectively, but that the converse inequality holds. We can therefore choose a sequence $\{(x_n, y_n)\}$ such that $(x_n, y_n) \to (\tilde{x}, \tilde{y})$ as $n \to \infty$ but, for every n,

$$|u(x_n, y_n) - u(x_n, \tilde{y})| \geq \varepsilon. \tag{20}$$

Choose $\tilde{z} \in Z$. As $f(x, y, z) = u(x, y) + v(y, z)$ is continuous, there exists N such that for all $n \geq N$ we have

$$|f(x_n, y_n, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z})| < \varepsilon/10. \tag{21}$$

Also, as $u(\cdot, y)$ is continuous for every y, $u(x_n, \tilde{y}) \to u(\tilde{x}, \tilde{y})$ as $n \to \infty$ and we can assume that for all $n \geq N$ we also have

$$|u(x_n, \tilde{y}) - u(\tilde{x}, \tilde{y})| < \varepsilon/10. \tag{22}$$

Consider

$$\begin{aligned}
&|f(x_n, y_n, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z})| \\
&= |u(x_n, y_n) + v(y_n, \tilde{z}) - u(\tilde{x}, \tilde{y}) - v(\tilde{y}, \tilde{z})| \\
&= \left|\left[u(x_n, y_n) - u(x_n, \tilde{y})\right] + \left[u(x_n, \tilde{y}) - u(\tilde{x}, \tilde{y})\right] + \left[v(y_n, \tilde{z}) - v(\tilde{y}, \tilde{z})\right]\right|.
\end{aligned}$$

By (22) we know that the middle square brackets denote a small expression, as does the sum of the three square brackets (by (21)). However, the first square brackets is at least ε by (20). This means that the last expression should also be relatively large. Specifically, for every n it follows that

$$|v(y_n, \tilde{z}) - v(\tilde{y}, \tilde{z})| \geq \varepsilon/2. \tag{23}$$

Consider now the sequence $\{(x_0, y_n, \tilde{z})\}_n$ and observe that $(x_0, y_n, \tilde{z}) \to (x_0, \tilde{y}, \tilde{z})$ as $n \to \infty$. By continuity of f we should have

$$u(x_0, y_n) + v(y_n, \tilde{z}) \to u(x_0, \tilde{y}) + v(\tilde{y}, \tilde{z}).$$

However, $u(x_0, y_n) = u(x_0, \tilde{y}) = 0$[17] while (23) shows that $v(y_n, \tilde{z})$ fails to converge to $v(\tilde{y}, \tilde{z})$, which is a contradiction. $\square$

We finally show that u is continuous. Assume that $(x_n, y_n) \to (x, y)$ as $n \to \infty$ for some point $(x, y) \in X \times Y$. Writing

$$\begin{aligned}
&u(x, y) - u(x_n, y_n) \\
&= \left[u(x, y) - u(x_n, y)\right] + \left[u(x_n, y) - u(x_n, y_n)\right],
\end{aligned}$$

we observe that the first brackets converges to 0 because $u(\cdot, y)$ is continuous (for each y separately) and the second one converges to 0 because of Lemma 12. $\square$

6.4.4. Uniqueness

Given a representation by u, v, it is straightforward that

$$\begin{aligned}
u'(x, y) &= \alpha u(x, y) + \beta(y) \\
v'(y, z) &= \alpha v(y, z) - \beta(y) + \gamma
\end{aligned} \tag{24}$$

also represent preferences for every $\alpha > 0$, a continuous function $\beta : Y \to \mathbb{R}$ and $\gamma \in \mathbb{R}$.

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[17] Observe that the crucial fact is only that $u(x_0, \cdot)$ is continuous, while in our construction it was guaranteed to be constant.

Conversely, the construction of the functions showed that, given the normalization $u(x_0, \cdot) = 0$, the only degrees of freedom left are multiplication of both u and v by a positive constant and shifting of v by an additive one.

However, relaxing the constraint $u(x_0, \cdot) = 0$, one can replace it by any continuous function β so that $u(x_0, \cdot) = \beta(y)$. Conversely, denoting $\beta(y) = u(x_0, \cdot)$ one observes that the transformation (24) holds.

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