Consumption of Values*

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Abstract

Consumption decisions are partly influenced by values and ideologies. Consumers care about global warming, child labor, fair trade, etc. We develop an axiomatic model of intrinsic values – those that are carriers of meaning in and of themselves – and argue that they often introduce discontinuities near zero. For example, a vegetarian's preferences would be discontinuous near zero amount of animal meat. We distinguish intrinsic values from instrumental ones, which are means rather than ends and serve as proxies for intrinsic values. We illustrate the relevance of our value-based model in different contexts, including equity concerns and prosocial behavior.

1 Introduction

In November 2015 Volkswagen sales in the US were about 25% lower than the year before. This dramatic drop followed a notice by the United States Environmental Protection Agency about the car manufacturer's violation of the Clean Air Act. It stands to reason that consumers were reacting to the facts that Volkswagen was selling cars that polluted

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the air beyond the allowed limits, and was also deceitful about it. These consumption choices were, at least partly, determined by values that were compromised by the firm's conduct: minimizing pollution and being honest.

Along similar lines, Nike has been struggling with information and rumors about its production practices for decades. In the 1990s it was reported that the company had been using sweatshops and child labor. Nike made a major effort to clean up its image, in an attempt to avoid the negative impact on sales. Analogously to the Volkswagen's case, Nike's behavior had to do with what consumers perceived as the right choice: using child labor is considered immoral.[1]

These are but two examples in which consumers care not only about the product they get for their money, but also about values, and, in particular, about potential conflict between their consumption and values they hold. Many consumption decisions are affected by the degree to which the production and/or the consumption processes hurt wildlife and endangered species, the globe and sustainability of life on it, or help underprivileged populations, promote equality, and so forth. For example, De Pelsmacker, Driesen, and Rayp (2005) found that consumers expressed a higher willingness to pay for coffee that was labeled "Fair Trade", while Hainmueller, Hiscox, and Sequeira (2015) showed that the label increased market share in a field experiment. Such ethical concerns affect firms' decisions as well as their profitability (see, e.g., Servaes and Tamayo, 2013). Indeed, the concept of Corporate Social Responsibility (CSR) might be partly a response to consumers' demand for values.[2]

This paper develops an axiomatic model of consumer choice where consumers derive utility not only from material bundles, but also from values. Specifically, we focus on principles, which are to be thought of as binary values: they are either respected or violated. Incorporating principles (and, more generally, values) into the consumer's utility function calls into question basic properties of consumer preferences: continuity and monotonicity. The following three examples shed light on these key aspects.

Example 1: Mary declares she is vegetarian. It is a matter of principle for her not to consume animals' meat. The amount of meat used in a product is immaterial to her; the very fact that it exists at a positive level is distinctly different from non-existence.

[1]Nike argued that it had no control over the practices employed by its sub-contractors. We make no claim about Nike's actual conduct in this case, nor about Volkswagen's in the previous one. We only point out that consumers seem to care about values, and perceived disrespect for values can affect consumption choices.

[2]See Garriga and Mele (2004) for a survey of CSR theories.

Obviously, she cannot tell whether a dish contains a minuscule amount of meat, but she attaches meaning to the act of consumption, and knowing that the dish contains some positive amount is sufficient to change the meaning of consumption.

Meaning introduces discontinuity at zero quantity, as well as violations of monotonicity since increasing the amount of meat in Mary's bundle will lower her utility. Such preferences can be captured by maximization of

$$U(x) = \begin{cases} u(x) & x \text{ is vegetarian} \\ u(x) - \gamma & \text{otherwise} \end{cases} \tag{1}$$

where $u(x)$ is a continuous function that measures the consumer's hedonic utility, and $\gamma > 0$ measures the degree to which she cares about vegetarianism. When γ is moderately large, maximization of U can capture the behavior of a vegetarian who would consume meat rather than die of hunger, but who would not do so for the sake of sheer pleasure.

Example 2: John writes his will, leaving his estate to his two children. He would like each of the children to have as much property as possible, but it is also important to him to have an equal division between them. This requires selling some assets, incurring transaction costs. John tells his lawyer that, as long as the costs do not exceed 5% of the estate, he prefers to incur them for the sake of an equal division.

John's preferences also exhibit discontinuity along a specific subspace: if we denote the two children's shares of the estate by $(y_1, y_2) \in [0,1]^2$, John assigns a special value to the subspace $y_1 = y_2$. Moreover, he might prefer the point $(0.48, 0.48)$ to $(0.49, 0.51)$, violating monotonicity. We can capture his preferences for equality by the function

$$U(y_1, y_2) = \begin{cases} u(y_1, y_2) & y_1 = y_2 \\ u(y_1, y_2) - \gamma & \text{otherwise} \end{cases} \tag{2}$$

where $u(y_1, y_2)$ is a continuous symmetric function and $\gamma > 0$ is the weight attached to the principle of equality.[3]

Example 3 (Gneezy and Rustichini, 2000a, 2000b): A school faces a problem: some parents are consistently late in picking up their children at the end of the day. In an attempt to deal with this issue, the school decides to impose a small fine on latecomers, and it turns out that the number of late parents unexpectedly grows larger. The explanation suggested

[3] While Ben-Porath and Gilboa (1994) and Fehr and Schmidt (1999) treat inequality continuously, the present formulation allows for discontinuity, conceptualizing equality as an intrinsic principle.

by Gneezy and Rustichini is that the fine allowed parents to be late without feeling guilty about it: once the fine is instated, it becomes a price; the school's after-duty services turn into a tradable good. Importantly, any positive amount of money can relieve a parent from guilt feelings, because any such amount would change the meaning of being late. Formally, if we denote by $t \geq 0$ the duration of the delay and by $m \geq 0$ – the fine paid, we can think of the parent as maximizing a utility function

$$U(t,m) = \begin{cases} u(t,0) - g(t) & m = 0 \\ u(t,m) & m > 0 \end{cases} \tag{3}$$

The function $u(t,m)$ is a continuous function that measures the parent's hedonic well-being, irrespective of potential guilt feelings. If the parent values both flexibility and money, $u(t,m)$ would be increasing in t and decreasing in m. The function $g(t)$ measures the extent of guilt, and we may assume that it is increasing in t, with $g(0) = 0$. When a fine $m > 0$ is imposed, guilt disappears, and $g(t)$ is not deducted from $u(t,m)$. Consider t with $g(t) > 0$. For $m \searrow 0$, we have $U(t,m) \to u(t,0) > U(t,0)$, violating continuity; further, for a small $m > 0$, $U(t,m) > U(t,0)$, violating monotonicity. Consequently, when the fine m is small, some parents may find it optimal to be late by some $t > 0$, even if they weren't late when there was no fine ($m = 0$), due to the emotional cost $g(t) > 0$. The school's optimal policy should therefore be either to impose a large enough fine or not at all.[4]

These examples involve agents who wish to abide by a certain principle (vegetarianism, equity, being on time and/or not to burden other people, respectively), and, as a result, they exhibit violations of monotonicity and of continuity. To understand why, let us go back to the standard justifications of these assumptions. The standard rationale for monotonicity is free disposal: a consumer need not physically consume products that she legally owns. But in the presence of values free disposal no longer holds. A person might feel guilty about the degree to which the bundle she owns hurts certain causes. Because there is no free disposal of emotions, preferences need not be monotone.[5]

[4] Note that our scope here is to show how our value-based approach can account for the discontinuities patterns observed in the motivation crowding-out literature. The guilt function $g(t)$ does not need to be discontinuous at $t = 0$. We refer to Noor and Ren (2023) for a model of guilt with continuous utilities in a menu-choice setup.

[5] One may argue that a principle is, by definition, something for which the agent is willing to give up hedonic well-being. This, however, does not necessarily imply violation of monotonicity, because an increase in consumption quantities x may lead to an increase in hedonic well-being ($u(x)$) that is enough to offset the negative impact this consumption has on principles.

The standard justification of continuity is human physiology: minuscule amounts cannot be discerned by our senses and therefore cannot have a noticeable effect on choices. However, when principles are concerned, the *meaning* of choice introduces discontinuity. Clearly, agents need to be aware of the quantities involved: Mary is assumed to know if a dish contains meat, John explicitly writes the proportions of the estates he bequeaths, the parent is aware of having to pay a fine. Given such awareness, discontinuity may arise from the assignment of meaning to numerical quantities.

Weber (1922) distinguished between intrinsic values, which are carriers of meaning in and of themselves, and instrumental values, which attain meaning only via some other mechanism that affects intrinsic values. Consider, for example, minimizing carbon dioxide emissions. Most consumers do not attach any deep meaning to the location of CO_2 molecules in the atmosphere in and of itself. But emission causes global warming, which, in turn, reduces people's well-being in many ways, and the latter is something that people care about intrinsically. Thus, minimizing emissions is an instrumental value: it needs to set in motion some mechanism in order to attain meaning. Because such a mechanism tends to be continuous, we believe that intrinsic values are more likely to give rise to discontinuities than are instrumental values.[6] Be that as it may, we are interested in values that result in discontinuity of preferences.

We focus on the simplest model of intrinsic principles and provide an axiomatic derivation of consumer preferences that can be described by a function U as in (1), (2), and (3). From a theoretical viewpoint, the axiomatic foundation helps us determine the most appropriate functional form in order to model the discontinuity introduced by an intrinsic principle. From an empirical/experimental viewpoint, our model can be utilized to test whether a consumer wishes to abide by a certain principle and whether such a principle is subjectively perceived as intrinsic or not.

The axiomatization of preferences as in (1), (2), and (3) is provided in Section 2. It turns out that (with only one principle), little needs to be assumed to obtain this additively separable representation, and, somewhat surprisingly, we find a unifying behavioral foundation for seemingly different phenomena. A survey of related literature is provided in Section 3. Section 4 concludes with a discussion about some useful extensions of our model. All proofs are contained in the Appendix. An Online Appendix contains an auxiliary result and examples proving the independence of our key axioms.

[6] One may consider models of intrinsic values that are continuous in quantities, as in Benabou and Tirole (2003, 2006).

2 The Model

2.1 Set-up

The alternatives are consumption bundles in X, which is a closed and convex subset of $\mathbb{R}^n_+$. For each good $i \le n$ there is an indicator $d_i \in \{0,1\}$ denoting whether the good violates the principle. That is, $d_i = 1$ implies that the good is inconsistent with the principle (say, contains meat), and $d_i = 0$ – that it doesn't (purely vegetarian). The consumer is aware of the vector $d \in \{0,1\}^n$, where we assume that producers should and do truthfully disclose the ingredients of their products. Thus, we assume that d is observable and verifiable, both to the consumer and to an outside observer.[7]

We wish to axiomatize the model in which, given d, the consumer maximizes $U(x) = u(x) - \gamma \mathbf{1}_{\{d \cdot x > 0\}}$where $d \cdot x$ is the inner product of the two vectors, so that $d \cdot x > 0$ if and only if there exists a product i that violates the principle ($d_i = 1$) and that is consumed at a positive quantity in x.

In this paper we assume that the vector d is known and kept fixed. That is, the consumer is provided with information about the goods that are and are not vegetarian, and we implicitly assume that this information is trusted.[8] We keep the information fixed, and can therefore suppress d from the notation, assuming that a binary relation $\succsim_d = \succsim \subset X \times X$ is observable. The information contained in the vector d is summarized by the answer to the question, is $d \cdot x > 0$? We thus define $X^0 = \{ x \in X \mid d \cdot x = 0 \}$, that is, all consumption bundles that do not use any positive amount of the "forbidden" goods, while $X^1 = X \backslash X^0 = \{ x \in X \mid d \cdot x > 0 \}$ contains the other bundles. Observe that X^0 is closed and convex and X^1 is convex.

Before moving on, we introduce some notation. The term "a sequence $(x_n)_{n \ge 1} \to_{n \to \infty} x$" will refer to a sequence $(x_n)_{n \ge 1}$ such that $x_n \in X$ for all n, and $x_n \to_{n \to \infty} x$ in the standard topology, where $x \in X$. When no ambiguity is involved, we will omit the index notation "$n \to \infty$" as well as the subscript "$n \ge 1$". We will use the notation "a sequence $(x_n) \subset A$" for "a sequence $(x_n)_{n \ge 1}$ such that $\{x_n\}_{n \ge 1} \subset A$". Conditions that involve an unspecified

[7] One may consider a model in which the vector d is not observable, but rather inferred from choice. For example, we may assume that violations of monotonicity can only be caused by value considerations, and identify the case $d_i = 1$ if there is an instance of preferences of a bundle with $x_i = 0$ to the same bundle with $x_i > 0$.

[8] One may extend the model to allow for the possibility that d is not reported. This can capture a wider range of phenomena. For example, an agent who is about to take a flight might not be thinking about its environmental effects. Once airlines start reporting the environmental damage per flight (d_i) – the agent may suddenly be aware of the value-effect of her consumption decisions, and perhaps change them.

index such as $x_n \succsim y_n$ are understood to use a universal quantifier ("for all $n \geq 1$"). Finally, when no confusion is likely to arise we will also omit the parentheses and use $x_n \to x$ rather than $(x_n) \to x$.

2.2 Axioms

We impose the following axioms on $\succsim$. We start with the standard assumption positing that choice behavior is described by a complete preorder.

A1. Weak Order: $\succsim$ is complete and transitive on X.

The next axioms will make use of the following key notion:

Definition 1 *Two sequences $x_n \to x$ and $y_n \to y$ are* comparable *if*
(A) there exist $i, j \in \{0, 1\}$ such that $(x_n) \subset X^i, x \in X^i$ and $(y_n) \subset X^j, y \in X^j$
or
(B) there exist $i, j \in \{0, 1\}$ such that $(x_n), (y_n) \subset X^i$ and $x, y \in X^j$.

Clearly, if all of the elements of $(x_n), (y_n)$, as well as the limit point of each are in the same subspace – X^0 or X^1 – the sequences are comparable.[9] However, two sequences $x_n \to x$ and $y_n \to y$ are comparable also in two other cases: first, (A) if (x_n) as well as *its* limit x are all in one subspace, while (y_n) with *its* limit, y, are all in another. And, second, (B) if the elements of both sequences belong to X^1 and the limits of both belong to X^0. (In principle, the opposite is also allowed by the definition, but X^0 is closed, so we cannot have a sequence in it converging to a point in X^1.) Basically, comparability rules out cases in which the transition to the limit makes only one sequence cross the boundary between the subspaces, leaving X^1 and reaching X^0. If this occurs, then the information we gather from preferences along the sequences is not very useful for making inferences about the limits: one sequence changes in a way that is discontinuous, and the other one doesn't. (See Fig. 1.)

By contrast, if the two sequences are comparable because none of them crosses the boundary between the two subspaces, then there is no reason for any violation of continuity. And, importantly, if both do cross the boundary, we still expect preference information along the sequences (where both (x_n) and (y_n) are in one subspace, which can only be X^1 in this case) to carry over to the limits (even though these are located in another subspace).

We can now state our continuity axiom:

[9] Here and in the sequel we use the terms "space" and "subspace" in the topological sense.

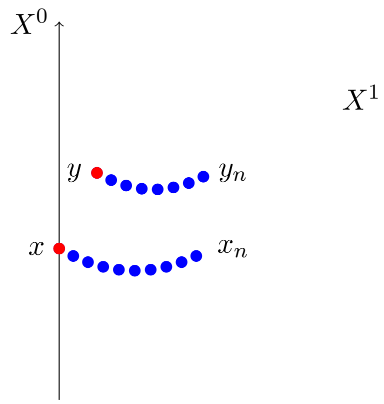

Figure 1: The comparability notion rules out this case.

A2. Weak Preference Continuity: For all comparable sequences $x_n \to x$ and $y_n \to y$, if $x_n \succsim y_n$ for all n, then $x \succsim y$.

Observe that, without the comparability condition, A2 would be a standard, though rather strong axiom of continuity: it would simply say that the graph of the relation $\succsim$ is closed in $X \times X$. This axiom is stronger than the standard continuity axiom of consumer choice, though it is implied by it when the relation $\succsim$ is also known to be a weak order. In our case, however, the consequent of the axiom is only required to hold if the sequences are comparable. As explained above, $x_n \succsim y_n$ for all n may not imply $x \succsim y$ (in the limit) if, for example, y is the only element involved that is in X^0; in this case it can enjoy the extra utility derived from obeying the principle, and thus $y \succ x$ can occur at the limit with no hint of this preference emerging along the sequence.

Clearly, if we restrict attention to one subspace, that is, if all of (x_n), (y_n), x, y are in X^1 or if all of them are in X^0, we obtain a standard continuity condition. Indeed, this would suffice to represent $\succsim$ on X^0 by a continuous utility function u^0 and to represent it on X^1 by a continuous utility function u^1, where u^0 and u^1 (having disjoint domains) need not have anything in common.

While A2 deals with weak preferences that are carried over to the limit, we will also need a corresponding axiom for strict preferences:

A3. Strict Preference Continuity: For all comparable sequences $x_n \to x$ and $y_n \to y$, and all $z, w \in X$, if $x_n \succsim z \succ w \succsim y_n$ for all n, then $x \succ y$.

To see the meaning of this axiom, assume, again, that comparability were not required. In this case, $x_n \succsim z$ and $w \succsim y_n$ would imply $x \succsim z$ and $w \succsim y$, respectively, and from $z \succ w$ we would easily conclude $x \succ y$. In our case, however, we could have that $(x_n) \subset X^1$ and

$x \in X^0$, and thus we cannot conclude that $x \gtrsim z$ (and, naturally, the same holds for w and y). Yet, comparability of $x_n \to x$ and $y_n \to y$ suffices to conclude that the preference gap between z and w is indeed enough to guarantee a strict preference between x and y. It is worthy of note that our A3 implies the "Cauchy continuity" property as defined in Kopylov (2016) in the context of providing extension results of continuous preference representations.[10]

Next, we introduce an Archimedean axiom stating that the "cost" of the principle in terms of utility is strictly positive, and, moreover, that no utility difference over X^0 exceeds infinitely many such "costs". Specifically, consider a sequence $(z_n) \subset X^1$ that converges to a point $z \in X^0$. In terms of hedonic utility, the bundles z_n become practically indistinguishable from z. However, the fact that z satisfies the principle means that its overall utility is higher than the limit of the corresponding utility values along the sequence. Intuitively, reaching X^0 at the limit provides an extra utility boost, which is not captured by the (continuous) hedonic utility, but should be captured in our overall-utility representation. One way to see this in terms of preferences is the following: if, along the sequence, $z_n \sim y \in X^0$, then we should have strict preference at the limit, $z \succ y$. In this case, the (hedonic) utility gap between z and y is a measure of the contribution of the principle to overall utility. The axiom states that, when aggregated, these measures are large enough to cover the entire utility range over X.

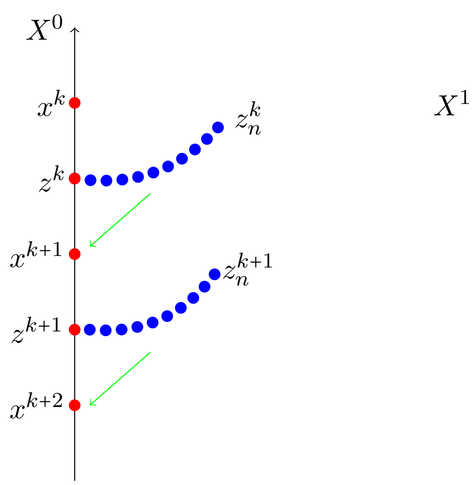

Figure 2: An illustration of A4

That is, an infinite chain of such preference gaps in X^0—as shown in Fig. 2, $x^k \succ x^{k+1} \succ$

[10]This condition is weaker than our A3, and it is not sufficient to derive our desired result. See the Online Appendix for further discussion.

$x^{k+2} \succ \ldots$— cannot be bounded above or below by some bundle $\hat{x}$ (otherwise $\hat{x}$ would take an infinite utility value).

Explicitly,

A4 Archimedeanity: Let $\left(x^k, z^k\right) \subset X^0$ and $\left(z_n^k\right)_{n,k\geq 1} \subset X^1$ be such that (i) $z_n^k \to z^k$, (ii) $x^k \succsim z^k$ and (iii) $z_n^k \succsim x^{k+1}$ for all $k \geq 1$ ($z_n^k \succsim x^{k-1}$ for $k \geq 2$) and for all $n \geq 1$. Then there does not exist $\hat{x} \in X$ such that $x^k \succsim \hat{x}$ ($\hat{x} \succsim x^k$) for all $k \geq 1$.

Axiom A4 rules out the case in which the agent prefers any bundle that violates the principle to any bundle that respects it. It thus applies to agents who care about not violating the principle, at least in some instances. To see how Axiom A4 entails the desirability of the principle, note that it rules out the case in which the sequence $\left(x^k\right)_{k=1}^{\infty}$ is constant at a bundle x, as x itself would bound the sequence from above and below. Formally, it means that for any sequence $x_n \to x$ with $(x_n)_{n=1}^{\infty} \subset X^1$ and $x \in X^0$, we must have $x \succ x_n$ for all large enough n. Indeed, if $x_n \succsim x$ for all n, we could set $\hat{x} = x^k = z^k = x$ and $z_n^k = x_n$ for all k, n, and the constant sequence $\left(x^k = x\right)_{k=1}^{\infty}$ with $\hat{x} = x$ would violate A4. More generally, A4 implies a natural discontinuity property:

> **Discontinuity**: Let $x, y \in X^0$, and let there be a sequence $x_n \to x$ with $(x_n)_{n=1}^{\infty} \subset X^1$ such that $x_n \succsim y$. Then $x \succ y$.

This is because $y \succsim x$ implies $x_n \succcurlyeq x$, which, by the argument above, violates A4. Indeed, the hedonic gap between x and y should be at least the cost of principle. By ruling out bounded yet infinite such gaps, A4 ensures that the utility value of any bundle in X be finite, and that the hedonic difference between any two alternatives in X can always be measured in terms of finitely many "costs" of the principle.[11]

Finally, we find it convenient to rule out the case in which all points in X^0 are equivalent.

A5 Non-Triviality: There are $x, y \in X^0$ such that $x \succ y$.

2.3 Representation Result

We are now ready to state our behavioral characterization of preferences that satisfy the aforementioned axioms.

[11] In Appendix B.2 we show that axiom A4 is implied by an intuitive (Lipschitz) continuity condition that requires the bundles x and y as above to be not too close.

Theorem 1 *For a given $d \in \{0,1\}^n$, the relation $\succsim$ on X satisfies A1-A5 if and only if there exist a continuous function $u : X \to \mathbb{R}$, which isn't constant on X^0, and a constant $\gamma > 0$ such that $\succsim$ is represented by*

$$U(x) = u(x) - \gamma \mathbf{1}_{\{d \cdot x > 0\}} \tag{4}$$

Our preference characterization can be viewed as a minimal departure from standard utility theory, aimed at explicitly modeling the influence of intrinsic principles on decision-making and examining how utility changes in response to value-relevant information. Despite its minimalist nature, the preference representation in (4) is sufficiently flexible to encompass several value-related expressions of preferences. Notably, all the examples presented in the Introduction, despite their differences, fall within the scope of our model, as shown next.

Example 1 directly embodies our functional specification, with the vector $d \in \{0,1\}^n$ representing the principle of vegetarianism. Mary categorizes bundles into two groups: X^0 is the space of all bundles x that do not contain any amount of meat as indicated by $d \cdot x = 0$; while X^1 contains the bundles that violate the principle. In the latter case, Mary anticipates a psychological cost γ associated with consuming the bundle which is subtracted from the hedonic utility.

There are situations in which a principle is satisfied on a subspace of alternatives which is not necessarily on the boundary of the entire space. This is demonstrated in Example 2 on egalitarianism, where John is expected to violate continuity near the diagonal $y_1 = y_2$. Example 2 can fit into our setup by changing the variables, exploiting the symmetry assumption. Specifically, we can set $x_1 = \max(y_1, y_2)$ and $x_2 = \max(y_1, y_2) - \min(y_1, y_2)$ and allow $d = (0,1)$ to embody the equality principle.[12]

As for Example 3, let us focus on situations where parents are late, excluding the non-interesting case where $t = 0$. We can divide the set of pairs (t, m) into two subsets: $X^0 = \{(t,m) : t > 0, m = 0\}$, which corresponds to situations where the parent is late but does not pay a fine for that, and $X^1 = \{(t,m) : t > 0, m > 0\}$, where a positive fine is imposed. In this simplified setup, $d = 1$ captures the principle of accepting to be late in exchange for a fine. As shown in Example 3, the agent is willing to pay a small fine if it

[12] In this example, the space of unequal divisions X^1 is not convex. Yet, it is the union of two convex sets. This means that a similar representation result would hold if we define the utility on each subspace of X^1 and impose an additional symmetry axiom.

alleviates the guilt she would feel otherwise. Our model then captures the discontinuity at $m = 0$ observed in Gneezy and Rustichini (2000a, 2000b). There are two important distinctions between this example and the previous ones. First, in the former we allow guilt $g(t)$ to be weakly increasing in the amount of delay t, while in the latter, as in our representation, γ is constant (independent of t). This, however, turns out not to be a substantial point, because our result allows for a seemingly more general representation, where γ depends on t. We elaborate on this in subsection 2.3.1 below. Second, as opposed to the previous examples where satisfying the principle brings pleasure, Example 3 presents a symmetric situation: respecting the principle is actually costly due to the guilt incurred by the agent. Formally, in our axiomatic model, this corresponds to having $\gamma < 0$ and it can be accommodated by modifying axiom A4 accordingly.

Uniqueness To what extent is the representation unique? The answer depends on the range of u and on γ. For example, if $\gamma > \sup_{x\in X^1}(u(x)) - \inf_{x\in X^0}(u(x))$, we have $U(x) > U(y)$ for all $x \in X^0, y \in X^1$ and the consumer would never give up the principle. In this case the utility function is only ordinal: any monotone transformation of u and γ that satisfies the above inequality represents preferences, and the utility function is far from unique. If, by contrast, γ is very small relative to $\sup_{x\in X^1}(u(x)) - \inf_{x\in X^0}(u(x)) > 0$, the monotone transformations that respect the representation (4) are much more limited. As will be clear from the proof, u is ordinal until a point of equivalence between two bundles $x \in X^0, y \in X^1$, and then the utility is uniquely determined throughout the preference-overlap between X^0 and X^1. Clearly, shifting u by a constant and multiplying both u and γ by a positive constant is always possible. Thus, on the preference-overlap between X^0 and X^1 we have a cardinal representation, and outside this preference interval – only an ordinal one.[13]

Observability Axiomatic derivations are supposed to relate theoretical concepts to observations. If the latter term only refers to databases of choice instances, one can only have finitely many observations in one's database, and all continuity axioms become vacuous. What is the point, then, of axiomatizing discontinuous preferences? Can one ever tell the difference between our restricted continuity assumptions and the classical, unrestricted ones?

The answer is in the affirmative if one adopts a slightly more general definition of "an

[13]This is reminiscent of the degree of uniqueness of representations of a semi-order by a function u and a just-noticeable-difference $\delta > 0$. See, for instance, Beja and Gilboa (1992).

observation". There are at least two types of situations in which we may consider infinitely many binary choices. The first involves mind experiments: a consumer may imagine infinitely many choices – such as infinitely many bundles with varying amounts of meat. The second pertains to stated preferences. Individuals, households, and organizations often state their preferences in a natural or even formal language, and such statements can describe a preference relation that is defined between any pair of bundles taken out of an infinite set. Our analysis can be helpful in testing how reasonable are such descriptions of preferences.

For example, Rubinstein (1988) introduced the notion of definable preferences, and called for modeling preferences that can be described within a formal language. Specifically, the lexicographic order, which might appear as a mathematical anomaly when using calculus, is a rather natural example when preferences are stated in natural language. At the same time, the fact that there is no real-valued representation of lexicographic preferences might indicate that these preferences, though easily stated, may not be acted upon. By contrast, the statement of a vegetarian's preferences, which can be numerically represented as above, may be more convincing. Thus, our axiomatic study can be helpful in telling apart stated preferences that are more reasonable than others.

2.3.1 An equivalent formulation

As will be clear from the proof, one can state the main result somewhat differently: define a function $g : \mathbb{R} \to \mathbb{R}_+$ to be *Archimedean* if there does not exist an infinite sequence $(u_i)_{i \geq 1}$ such that $u_{i+1} - g(u_{i+1}) \geq u_i$ that is bounded from above.[14] Then our main result can be stated as follows:

Proposition 1 Let there be given $d \in \{0,1\}^n$. The following are equivalent:

(i) the relation $\succsim$ on X satisfies A1-A5;

(ii) there exist a continuous function $u : X \to \mathbb{R}$, which isn't constant on X^0, and a continuous Archimedean function $g : \mathbb{R} \to \mathbb{R}_+$ such that $\succsim$ is represented by u on X^0 and X^1 separately, and, as a whole on X, by

$$U(x) = u(x) - g(u(x))\,\mathbf{1}_{\{d \cdot x > 0\}};$$

(iii) there exist a continuous function $u : X \to \mathbb{R}$, which isn't constant on X^0, and a

[14] Clearly, if g is Archimedean, it is strictly positive.

constant $\gamma > 0$ such that $\succsim$ is represented by

$$U(x) = u(x) - \gamma \mathbf{1}_{\{d \cdot x > 0\}}.$$

Clearly, the constant function $g(u(x)) = \gamma$ is Archimedean for all $\gamma > 0$. Thus, the representation in (iii) is a special case of that in (ii). Our proof shows, however, that they are equivalent. In the main theorem we state only (iii), as it seems more parsimonious. Yet, for some applications, one may find the added flexibility of (ii) useful. This can happen if we expect the function u to provide more information than the mere ranking of bundles, or if we wish to exploit some special functional forms. For example, if utility differences are related to probabilities of choice (as in the stochastic choice literature), one cannot rescale u to obtain a constant g without losing information about choice probabilities. Alternatively, considering an application such as Example 3, we may wish to use an additively separable u, a structure than might be lost by a transformation that renders g constant.

3 Related Literature

Standard economic theory tends to ignore value considerations and conceptualize a consumer's utility as a function of her own bundle (e.g., Varian, 1978; Kreps, 1990; Mas-Colell-Whinston-Green, 1995). Yet, in a variety of domains ranging from applied economics to marketing, it has long been observed that consumers care about ethical values. Auger, Burke, Devinney, and Louviere (2003) and Prasad, Kimeldorf, Meyer, and Robinson (2004) find that consumers are conscientious and express willingness to pay more for products that have desirable social features, such as environmental protectionism, avoiding child labor, as well as sweatshops. Barnett, Cloke, Clarke, and Malpass (2005) discuss the notion of "consuming ethics".

Taking a broader perspective, the notion that consumption has socio-psychological effects has long been recognized (Veblen, 1899; Duesenberry, 1949). Frank (1985a, 1985b) highlights the role of social status, and, more recently, Heffetz (2011) studies the effects of conspicuous consumption empirically. Interdependent preferences are also at the core of Fehr and Schmidt's (1999) inequity aversion, Karni and Safra's (2002) sense of justice, as well as Ben-Porath and Gilboa's (1994) axiomatization of the Gini Index, and Maccheroni, Marinacci, and Rustichini's (2012, 2014) works on envy and pride. Conspicuous consumption can be viewed as dealing with meaning, reflecting on one's social standing and identity. Inequity aversion can similarly be conceived of as an attitude towards the principle of equality.

Medin, Schwartz, Blok, and Birnbaum (1999) argue against formal models because of the lack of attention to meaning and signification. According to their approach, decision theory lacks the *semantics* of decisions. Meaning is also related to narratives one can construct. Eliaz and Spiegler (2020) deal with narratives of causality, and Glazer and Rubinstein (2021) – with stories that are sequences of events.

Recent developments in behavioral economics suggest formal modeling of some related phenomena, such as the axiomatic models of Dillenberger and Sadowski (2012) on shame over selfish behavior and Evren and Minardi (2015) on warm-glow. These works are similar to ours in introducing ethical considerations into the utility function.

In comparison with the above diversified literature, our main contribution is to incorporate principles into microeconomic theory, in terms of a formal, axiomatically-based model of consumer choice. In practice, our foundation brings to the fore a natural experimental test to elicit whether a person cares intrinsically about a certain principle—discontinuity at zero quantities, as in the case of vegetarians who prefer not to consume any amount of animal meat. The discontinuity test sheds light on two novel aspects.

First, as explained previously (see paragraph on Observability), individuals and organizations may often state facts about their preferences. However, not all such statements are equally credible. Some descriptions of preferences may sound more convincing than others and some policies are more likely to be implemented than others. In particular, natural language may easily describe discontinuous preferences. For instance, lexicographic preferences can be easily described in natural language but may not be supported by actual behavior. From this viewpoint, our discontinuity test provides a mean to verify, at least at the level of thought experiment, whether individuals really mean what they claim.

Second, as our examples suggest, our approach is applicable to different contexts and could be viewed as a complementary method to experimentally investigate well-known phenomena, such as inequity aversion and motivation crowding-out, based on the estimation of a single parameter, γ.

4 Extensions

Clearly, the utility function axiomatized in this paper is special in two ways. First, it deals with a single source of principles. That is, it can describe the behavior of a vegetarian consumer, but should the latter also care about child labor or Fair Trade, we will have to consider more general functional forms, allowing for discontinuities at several subspaces and carefully bridging them in the style of Axioms A2 and A3.[15] Second, the meaning attached to consumption is dichotomous. For example, if our consumer prefers not to eat animals at all, but, should she have to, prefers to eat seafood than mammals, we will again find that (1) is too special to describe her preferences. For both reasons one might be interested in a functional form with several additive terms such as γ above. Importantly, the resulting model would still involve discontinuities in quantities.

A natural extension of the model would represent preferences by a function

$$U(x) = u(x) + v(d, x),$$

where $v(d, x)$ is the utility derived from principles. If all principles are instrumental, one could expect v to be a continuous function (in the quantities x and also in d if it is modeled as a vector of continuous coefficients). By contrast, we suggest that intrinsic principles tend to generate discontinuities at zero quantities.

We mention in passing that the distinction between intrinsic and instrumental principles may be important for consumer choice: instrumental principles are more negotiable than the intrinsic ones. For example, a consumer who cares about the emission of CO_2 because of its damage to the environment may compensate for her consumption of flights by donating money for planting of trees. However, a vegetarian consumer would be less likely to feel that consuming meat is fine as long as one donates some money to animal rights organization.

An important direction for future research is the nature of optimization in our model, especially if one makes the plausible assumption that the consumer cares about multiple values/principles, and that they may interact in non-trivial ways. In particular, discontinuity at zero can render the optimization problem NP-Hard, with the combinatorial aspect resulting from the choice of variables that are consumed at strictly positive quantities. For example, we can think of a utility function that allows the consumer to violate no more than k principles, and embed a SET COVER problem in the optimization of U.[16]

[15] For decision making under risk, Gilboa, Minardi, and Wang (2024) present a model where multiple principles can be "risked" simultaneously. The utilities therein are fully cardinal, as in the expected utility model.

[16] Specifically, consider the following problem. Let there be n goods and m needs. Each good violates one (distinct) principle, and $v(d, x)$ is zero as long as no more than k of the x_i's are strictly positive, but it is -1 otherwise. A quantity x_i of good $i \leq n$ satisfies a need $j \leq m$ to degree $\delta_{ij} x_i$ according to an incidence matrix $\delta_{ij} \in \{0, 1\}$. The function $u(x)$ is a truncated Cobb-Douglas function defined on the degrees of need

Appendix

A Proof's Sketch

We explain the logic of the proof of Theorem 1 in three steps.

Step 1: Axioms A1-A3 help us find functions that we can think of as the hedonic utility u: each is continuous throughout X and correctly represents preferences on X^0 and X^1, separately. We note that A1 and A2 trivially imply that one can find continuous representations of $\succsim$ on X^0 and on X^1, because on each of these A2 implies the standard continuity axiom. This, however, does not mean that there exists a function that is continuous on all of X and that represents $\succsim$ both on X^0 and on X^1 (separately). The Online Appendix is devoted to an auxiliary result (Theorem 2) stating that A3 is the missing link: it states that any bounded and continuous utility function that represents $\succsim$ on X^1 has a unique continuous extension to X^0, in such a way that the extension represents $\succsim$ also on X^0. We do not expect this extension to represent preferences across the two spaces because we know that discontinuities are to be observed between them.

Step 2: We establish the more general version of representation (4) where the penalty γ can vary with x. Let $\tilde{X}^0 = \{x \in X^0 : \exists x' \in X^1, x \sim x'\}$ denote the part of X^0 that "overlaps" with X^1 in preference. We can define a "boost function" $\gamma_u : \tilde{X}^0 \to \mathbb{R}$ by setting $\gamma_u(x) = u(x') - u(x)$ for all $x \in \tilde{X}^0$ and $x' \in X^1$ with $x \sim x'$. Axiom A4 guarantees γ_u to be strictly positive on $\tilde{X}^0$, and to satisfy $inf_{x \in \tilde{X}^0} \gamma_u(x) > 0$ when $\tilde{X}^0 \neq X^0$. And, when $\tilde{X}^0 = X^0$ and $inf_{x \in \tilde{X}^0} \gamma_u(x) = 0$, it must be that for every $x' \in X^1$ there must exist an $x \in X^0$ with $x \succcurlyeq x'$. We show that this γ_u is continuous and can be extended continuously to the whole X^0 when $\tilde{X}^0 \neq X^0$. With the extended γ_u, we then prove that the function

$$\tilde{u}(x) = \begin{cases} u + \gamma_u, & x \in X^0 \\ u, & x \in X^1 \end{cases}$$

---

satisfaction

$$u(x) = \max\left[\prod_{j \le m}\left(\sum_{i \le n} \delta_{ij} x_i\right), 0.5\right]$$

so that it obtains a positive value only if all needs are satisfied, each to a positive degree (and no more than k principles are violated). We may assume that all prices are zero or that income is very large. The function $U(x) = u(x) + v(d, x)$ can obtain a positive value if and only if the incidence matrix $\delta_{ij} \in \{0, 1\}$ contains no more than k rows i that cover all m columns.

To make the above well-defined, one has to agree on the language in which the utility function is described.

represents $\succsim$ on X^0 and X^1 together.

Step 3: We exploit the ordinality of standard utility representations and choose to "scale" the utility u in such a way that the boost function γ_u is a constant $\gamma > 0$. Fixing an $x \in X^0$, we use $\gamma_u(x) > 0$ as a fixed measure of the utility of the principle. We then use $\gamma_u(x)$ to define "steps" on X^0 that intuitively correspond to "better than... by exactly the utility of the principle", and we transforms u correspondingly so that it increases by the same amount for each such step. We finally extend u to all of X. Axiom A4 guarantees that this transformation of u can eventually cover the whole X.

B Proofs of Representation Results

It will be convenient to introduce the following definition of a binary relation P on X^0:

Definition 2 *For $x, y \in X^0$, we say that xPy if there exists $z \in X^0$ and a sequence $z_n \to z$ with $(z_n) \subset X^1$ such that $x \succsim z$ and $z_n \succsim y$.*

Observe that, if we had no discontinuity between X^0 and X^1, the relation P could be expected to be equal to $\succsim$: if xPy, the conditions $z_n \to z$ and $z_n \succsim y$ would simply imply that $z \succsim y$, and $x \succsim y$ would follow by transitivity. Conversely, if $x \succsim y$, one could expect an open neighborhood of x to contain points z_n such that $z_n \succsim y$ even though $z_n \in X^1$ (for example, monotonicity would insure that this is the case). However, in the presence of discontinuity between X^1 and X^0, this is no longer the case. As explained above in the context of A4, we should expect z to be strictly better than y; indeed, intuitively, "z should be better than y at least by the cost of the principle". And the same should hold for any $x \in X^0$ such that $x \succsim z$.

Using this definition, the Archimedean axiom can be written as follows.

A4 Archimedeanity (in P terms): Let $(x_n) \subset X^0$ be such that $x_{n+1}Px_n$ (x_nPx_{n+1}) for all $n \geq 1$. Then there does not exist $\hat{x} \in X$ such that $\hat{x} \succsim x_n$ ($x_n \succsim \hat{x}$) for all $n \geq 1$.

This new formulation of A4 is simply a re-statement of the axiom in terms of the relation P. We therefore do not re-name the axiom.

B.1 Proof of Theorem 1

The proof of necessity of the axioms is straightforward and therefore omitted. To prove sufficiency, recall that $\succsim$ is continuous on X^1, and thus there exists a continuous bounded function v that represents $\succsim$ on X^1. By Theorem 2 (Online Appendix) we extend v continuously to all of X so that it represents $\succsim$ on X^0 as well. We will construct a continuous function U on X^0 that represents $\succsim$ and that also represents P by γ differences. We start out with any continuous function that represents $\succsim$ on those $x \in X^0$ for which there are no $y \in X^0$ such that xPy, use the function $v(\cdot) + \gamma$ on that set, and extend it to the rest of X^0 while respecting the representation of P by γ differences. Any element of X^1 that has a $\succsim$-equivalent in X^0 will have to have the same U value, and we will show that the resulting function is continuous on X^1 as well. Moreover, we will show that the function so constructed has a constant "jump" of γ between any sequence in X^1 that converges to a limit in X^0. We then extend it to elements of X^1 which are strictly better or strictly worse than all elements of X^0.

Proof.

Lemma 1 For $x, y \in X^0$, if xPy then $x \succ y$.

Proof: Assume not. Then there is a sequence $z_n \to z$, $(z_n) \subset X^1$, $x \succsim z$, and $z_n \succsim y$ but $y \succsim x$. By transitivity of $\succsim$, we also get $z_n \succsim x$ and by definition of P (with the same sequence $z_n \to z$), we have xPx. Define $x_n = x \in X^0$ such that $x_{n+1}Px_n$ for all n and the sequence is bounded (by $\hat{x} \equiv x$), in violation of A4.

Lemma 2 For $x, y, w \in X^0$, if xPy then (i) $y \succsim w$ implies xPw, and (ii) $w \succsim x$ implies wPy.

Proof: Suppose that $x, y, z \in X^0$ and $(z_n) \subset X^1$ are given, such that $z_n \to z$, $x \succsim z$ and $z_n \succsim y$. In case (i), $z_n \succsim y \succsim w$ and by transitivity $z_n \succsim w$, thus xPw by definition of P. As for (ii), $w \succsim x$ and $x \succsim z$ imply $w \succsim z$ and the definition of P yields wPy.

Lemma 3 For $x, y \in X^0$, if xPy, then there exists $z \in X^0$ and a sequence (z_n) with $z_n \in X^1$ such that $z_n \to z$, $x \succsim z$ and $z_n \sim y$.

Proof: Assume that $x, y \in X^0$ satisfy xPy, and that $z \in X^0$ and (z_n) with $z_n \in X^1$ satisfy $z_n \to z$, $x \succsim z$ and $z_n \succsim y$. We argue that, for each n, there exists $\alpha_n \in (0,1]$ such that $w_n \equiv \alpha_n z_n + (1-\alpha_n) y \in X^1$ satisfies $w_n \sim y$. Indeed, if $z_n \sim y$ set $\alpha_n = 1$. Assume, then, $z_n \succ y$. If there exists $\beta \in (0,1)$ such that $y \succ \beta z_n + (1-\beta) y$ then $z_n \succ y \succ \beta z_n + (1-\beta) y$, with $z_n, \beta z_n + (1-\beta) y \in X^1$, and Lemma 7 yields the existence of a point on the interval $[\beta z_n + (1-\beta) y, z_n]$ that is indifferent to y; that point is in $[y, z_n]$ and we are done. If such a β does not exist, $\beta z_n + (1-\beta)\, y \succ y$ for all $\beta > 0$. Taking a subsequence $\beta_k \searrow 0$, with $\beta_k z_n + (1-\beta_k)\, y \to y$, we obtain yPy, in contradiction to Lemma 1.

Hence there are $\alpha_n \in (0,1]$ such that $w_n \equiv \alpha_n z_n + (1-\alpha_n)\, y \sim y$; observe that $w_n \in X^1$ because $\alpha_n > 0$. Choose a convergent subsequence of α_n, $\alpha_{n_k} \to \alpha^*$. Then $w_{n_k} \to w^* \equiv \alpha^* z + (1-\alpha^*)\, y \in X^0$. To show that $x \succsim w^*$, observe that $z_{n_k} \succsim w_{n_k}$ (because $z_{n_k} \succsim y$ and $w_{n_k} \sim y$), $z_{n_k} \to z$,$w_{n_k} \to w^*$, while $(z_{n_k})_k, (w_{n_k})_k \subset X^1$ and $z, w^* \in X^0$. Hence $(z_{n_k})_k \to z$ and $(w_{n_k})_k \to w^*$ are comparable and A2 yields $z \succsim w^*$ and $x \succsim w^*$ follows by transitivity.

For the rest of the appendix, we use the notation X_P^0 to refer to the set defined as

$$X_P^0 = \left\{ y \in X^0 \,\middle|\, \exists x \in X^0,\, xPy \right\}.$$

The strategy of the proof is to choose the continuous representation of $\succsim$ on X^0 derived from Theorem 2, take a monotone and continuous transformation thereof to obtain another representation, u, that satisfies

$$xPy \qquad \Leftrightarrow \qquad u(x) - u(y) \geq \gamma > 0$$

and then extend the function u to X^1. To this end, it will be useful to know some facts about continuous representations of $\succsim$ on X^0.

Lemma 4 Let there be given a continuous function $u : X^0 \to \mathbb{R}$ that represents $\succsim$ (on X^0). Let $y \in X_P^0$. Then there exists $\gamma(y) > 0$ such that, for every $x \in X^0$, xPy iff $u(x) - u(y) \geq \gamma(y)$. Furthermore, $\gamma(y)$ can be extended to all of X^0 so that $w \succsim y$ iff $u(w) + \gamma(w) \geq u(y) + \gamma(y)$ (for all $y, w \in X^0$).

Proof: Define $P_{y+} = \left\{ x \in X^0 \mid xPy \right\}$.

Case (a): Let us first consider $y \in X_P^0$ so that $P_{y+} \neq \emptyset$.

Consider $u(P_{y+}) = \left\{ u(x) \in u(X^0) \mid xPy \right\}$. By Lemma 1, $u(y) < a$ for all $a \in u(P_{y+})$. By Lemma 2, $u(P_{y+})$ is an interval. We wish to show that it contains its infimum. Let $a = \inf u(P_{y+})$. For $k \geq 1$, let $x^k \in X^0$ be such that $a \leq u\left(x^k\right) < a + \frac{1}{k}$. Because $x^k Py$, by Lemma 3, there exist (i) $z^k \in X^0$ and (ii) $\left(z_n^k\right)_{n\geq 1}$ with $z_n^k \in X^1$ such that $z_n^k \to z^k$, $x^k \succsim z^k$ and $z_n^k \sim y$. Hence, $\forall k, l, m, n$, $z_n^k \sim z_m^l (\sim y)$. Because $\left(z_n^k\right), \left(z_m^l\right) \subset X^1$ converge to $z^k, z^l \in X^0$ respectively, A2 implies $z^k \sim z^l$. This means that $u\left(z^k\right) = u\left(z^l\right)$ $\forall k, l$ and thus $u\left(z^k\right) = a$. Hence, $a = \min u(P_{y+})$ and $a > u(y)$. It remains to define $\gamma(y) = a - u(y) > 0$. For every y such that $P_{y+} \neq \emptyset$, $\gamma(y)$ is bounded from above (by $u(x) - u(y)$ for any

$x \in P_{y+}$). Observe that $\gamma(y)$ is uniquely defined $\forall y \in X^0_P$. We now show that $u + \gamma$ also represents $\succsim$ for alternatives y, w in this range.

In the construction above, $u(y) + \gamma(y) = \min u(P_{y+})$. If $w \succsim y$, Lemma 2 implies that $P_{w+} \subset P_{y+}$ and thus $\min u(P_{w+}) \geq \min u(P_{y+})$, so that $u(w) + \gamma(w) \geq u(y) + \gamma(y)$ follows. To see that the inequality is strict if $w \succ y$, let $x \in X^0$ be a $\succsim$-minimal element in P_{w+}, that is, $u(x) = u(w) + \gamma(w)$. We claim that there exists x' with $u(x') < u(x)$ such that $x'Py$ still holds (while $x'Pw$ doesn't). Because xPw, by Lemma 3 there exists $z \in X^0$ and a sequence $(z_n)_{n\geq 1}$ with $z_n \in X^1$ such that $z_n \to z$, $x \sim z$ and $z_n \sim w$ (and $x \sim z$ follows from the minimality of x). Hence, $z_n \succ y$. As in the proof of Lemma 3, for each z_n we can find $\alpha_n \in (0,1]$ such that $t_n \equiv \alpha_n z_n + (1-\alpha_n) y \in X^1$ satisfies $t_n \sim y$ (or else yPy would follow).

Taking a convergent subsequence of α_n, say $\alpha_{n_k} \to \alpha^*$, we have $t_{n_k} \to t^* \equiv \alpha^* z + (1-\alpha^*) y \in X^0$. We thus have two sequences $(z_{n_k}), (t_{n_k}) \subset X^1$, with $z_{n_k} \sim w \succ y \sim t_{n_k}$ and $z_{n_k} \to z, t_{n_k} \to t^*$ with $z, t^* \in X^0$. Observe that $(z_{n_k}) \to z, (t_{n_k}) \to t^*$ are comparable. Hence A3 implies that $z \succ t^*$. Thus $\exists x' \in X^0$ with $u(x') \in (u(t^*), u(z))$. As z (and x) was selected to have the lowest possible u in $u(P_{w+})$, $x'Pw$ doesn't hold, while $x'Py$ does.

Case (b): For $y \in X^0 \backslash X^0_P$ we set $\gamma(y)$ to be a constant, defined as follows. Let $\bar{u} = \sup_{z \in X^0_P} u(z)$. This sup may or may not be a max.[17] Define

$$\gamma(y) = \lim_{n\to\infty} \sup \left\{ \gamma(z) \;\middle|\; \bar{u} - \frac{1}{n} < u(z) \leq \bar{u} \right\}.$$

It is finite because $\gamma(y)$ is bounded from above by $(u(x) - \bar{u} + 1)$ $\forall x \in P_{z+}$. Because $\gamma(y)$ is constant for all $y \in X^0 \backslash X^0_P$, and u represents $\succsim$ for alternatives y, w in this range, so does $u + \gamma$. Next, observe that $\sup_{z \in X^0_P} [u(z) + \gamma(z)] = \sup_{z \in X^0} u(z)$ and, for $y \in X^0 \backslash X^0_P$ and $w \in X^0_P$ we have $u(y) + \gamma(y) \geq \sup_{z \in X^0} u(z) \geq u(w) + \gamma(w)$ and $u(y) + \gamma(y) > u(w) + \gamma(w)$. That is, the value $\sup_{z \in X^0} u(z)$ might be obtained by $u(\cdot) + \gamma(\cdot)$ on X^0 or on $X^0 \backslash X^0_P$ but not on both, so that $u + \gamma$ represents $\succsim$ on the entire range.

Lemma 5 Let there be given a continuous function $u : X^0 \to \mathbb{R}$ that represents $\succsim$ (on X^0). There exists a continuous function $\phi : u(X^0) \to \mathbb{R}$ such that, for every $x \in X^0, y \in X^0_P$, xPy iff $u(x) - u(y) \geq \phi(u(y))$ and $u(\cdot) + \phi(u(\cdot))$ also represents $\succsim$ on X^0.

[17] For example, for $n = 2$, $X = [0,10]^2$ and $d = (1,0)$ consider $u_1(x_1, x_2) = x_2 + x_1$ and $u_2(x_1, x_2) = x_2 + (x_1 - 1)^2$. In both cases define the relation by the function u_i and $\gamma = 1$. In the case of u_1 the relation P is a closed subset of $X^0 \times X^0$ an $\bar{u} = 9$ is the max of $u(z)$ over X^0_P, whereas for u_2 P isn't closed, and the point $(9,0)$ is not in X^0_P, leaving $\bar{u} = 9$ the sup of the utility in X^0_P.

Proof: Use Lemma 4 to define $\gamma : X^0 \to \mathbb{R}$ such that $u(\cdot) + \gamma(\cdot)$ represents $\succsim$ on X^0 and xPy iff $u(x) - u(y) \geq \gamma(y)$ whenever $y \in X_P^0$ as above. Observe that, $\forall y, w \in X^0$, we have $w \succsim y$ iff $u(w) + \gamma(w) \geq u(y) + \gamma(y)$. Hence $w \sim y$ implies $u(w) + \gamma(w) = u(y) + \gamma(y)$ and, since $u(w) = u(y)$ also holds in this case, $\gamma(w) = \gamma(y)$. It follows $\exists \phi : \mathbb{R} \to \mathbb{R}$ such that $\gamma(y) = \phi(u(y))$, uniquely defined for all values $\bar{u} = u(y)$ such that $y \in X_P^0$. To show that ϕ is continuous on that range, let there be given $\bar{u} \in range(u)$ and $\left(u^k\right)_{k\geq 1}$ so that $u^k \in range(u)$ and $u^k \to \bar{u}$ (as $k \to \infty$). If $\phi\left(u^k\right) \to \phi(\bar{u})$ fails to hold, there exists $\varepsilon > 0$ such that (i) there are infinitely many k's for which $\phi\left(u^k\right) < \phi(\bar{u}) - \varepsilon$ or (ii) there are infinitely many k's for which $\phi\left(u^k\right) > \phi(\bar{u}) + \varepsilon$.

In case (i), let $y \in u^{-1}(\bar{u})$ and $y^k \in u^{-1}\left(u^k\right)$ for k from some k_0 on (obviously, with $y \in X_P^0$ and $y^k \in X_P^0$ for all k). As u is continuous, we can also choose such a y and a corresponding sequence so that $y^k \to y$. Let $t, t' \in X^0$ be such that $u(y) + \phi(u(y)) = u(t) > u(t') > u\left(y^k\right) + \phi\left(u\left(y^k\right)\right)$ for all $k \geq k_0$, so that $t \succ t'$, tPy^k,$t'Py^k$ for all k, tPy but not $t'Py$. As tPy we can select a sequence $(z_n) \subset X^1$ with $z_n \to z \in X^0$, $t \succsim z$ and $z_n \sim y$. By the choice of t (as a u-minimal element such that tPy), $u(t) = u(z)$. As $t'Py^k$, there is, $\forall k$, a sequence $\left(w_n^k\right) \subset X^1$ such that $w_n^k \to w^k \in X^0$, $t' \succsim w^k$ and $w_n^k \sim y^k$. As above, select a convergent subsequence of the diagonal to get a sequence $(w_n^n) \subset X^1$ such that $w_n^n \to w \in X^0$, $t' \succsim w \in X^0$ and $w_n^n \sim y^n$. By transitivity, $z \sim t \succ t' \succsim w$. Observe that $z_n \to z$ and $w_n^n \to w$ are comparable, and we also have $z \succ w$. Use Lemma 10 (Online Appendix) for $y_n = y^n \to y$ and $x_n = x = y$. Because $x_n, y_n, x, y \in X^0$, $y_n \to y$ and $x_n \to y$ are also comparable. Lemma 10 implies $y \succ y$, a contradiction.

In case (ii) select $t, t' \in X^0$ be such that $u(y) + \phi(u(y)) = u(t) < u(t') < u\left(y^k\right) + \phi\left(u\left(y^k\right)\right)$ $\forall k \geq k_0$, so that $t' \succ t$, tPy and $t'Py$ hold, but tPy^k,$t'Py^k$ do not hold for any k. For each k, $\exists t^k$ such that $u\left(t^k\right) = u\left(y^k\right) + \phi\left(u\left(y^k\right)\right)$ (a u-minimal element satisfying t^kPy^k). Let $\left(z_n^k\right) \subset X^1$ be such that $z_n^k \to z^k \in X^0$, $t^k \succsim z^k$ and $z_n^k \sim y^k$. Let $(z_n) \subset X^1$ be such that $z_n \to z \in X^0$, $t \succsim z$ and $z_n \sim y$. By the choice of t,$\left(t^k\right)$ as minimal elements, $t \sim z$ and $t^k \sim z^k$. Select a convergent subsequence of $z_k^k \to z^* \in X^0$. Because $z^k \succsim t'$ (and $z^k \in X^0$) we have $z^* \succsim t' \succ t$. Again, contradiction follows from Lemma 10.

To complete the proof, use Theorem 2 (Online Appendix) to have a continuous and bounded function $v : X \to \mathbb{R}$ that represents $\succsim$ on X^0 and on X^1. By A5, it isn't constant on X^0. Assume, w.l.o.g., that $\inf_{x\in X^0} v(x) = 0$ and $\sup_{x\in X^0} v(x) = 1$. Let $b = \inf_{x\in X} v(x)$ and $a = \sup_{x\in X} v(x)$ so that $b \leq 0 \leq 1 \leq a$. Next, define a continuous $u : X \to \mathbb{R}$ and $\gamma > 0$ such that $U(x) = u(x) - \gamma \mathbf{1}_{\{x\in X^1\}}$ represents $\succsim$. We first define $U = u$ on X^0, and $\Delta > 0$ such that u represents $\succsim$ on X^0, and (u, Δ) jointly represent P on X^0 by $[xPy \Leftrightarrow u(x) - u(y) \geq \Delta]$, and then define u on X^1 and γ.

Step 1: Definition of $U = u$ on X^0

If $P = \varnothing$ define $u = v$ and $\Delta = 2$. Clearly, the representation of P holds. Otherwise, if $P \neq \varnothing$, we construct a partition of X^0 into countably many subsets X_k^0 for $k \in \mathbb{Z}$ such that, if $x \in X_k^0$ and $y \in X_l^0$, then $k > l+1$ implies xPy and $k \leq l$ implies $\neg(xPy)$. First, we define a function $S : X^0 \times X^0 \to \mathbb{Z}$ to be the maximal k such that there are $z_0 = x, z_k = y, z_i P z_{i+1}$ for $\forall i \leq k-1$. For $x \succsim y \succsim z$, we have $S(z,y) + S(y,x) \leq S(z,x) \leq S(z,y) + S(y,x) + 1$. For $x, y \in X^0$ with $y \succ x$, set $S(y,x) = -S(x,y) - 1$ so that $S(y,x) + S(x,y) = -1$ for all $x \nsim y$. We finally define u on X^0. Distinguish between two cases:

Case 1a: $\forall x \in X^0$ $\exists y \in X^0$ such that xPy. In this case, should the representation of P by Δ hold, u should be unbounded from below. Select an $x_0 \in X^0$ with $P_{x_0+} \neq \varnothing$. For $k \in \mathbb{Z}$, let $X_k^0 = \left\{ y \in X^0 \mid S(x_0, y) = k \right\}$. For $y \in X_0^0$ (that is, $y \succsim x_0$ but not yPx_0), set $u(y) = v(y) - v(x_0)$ (in particular, $u(x_0) = 0$). Let $\Delta = \sup_{X_0^0} u(y)$. By Lemma 4, $\Delta > 0$. Once u is defined for all $y \in X_k^0$ for $k \geq 0$, extend it to X_{k+1}^0 as follows: $\forall x \in X_{k+1}^0$ $\exists y \in X_k^0$ such that $v(x) = v(y) + \phi(v(y))$ where ϕ is the function constructed in Lemma 5 for v (and by Lemma 5, this is the highest y that satisfies xPy). Set $u(x) = u(y) + \Delta$. Similarly, if u is defined for all $y \in X_k^0$ for $k \leq 0$, extend it to X_{k-1}^0 by $u(x) = u(y) - \Delta$ for $x \in X_{k-1}^0$ and $y \in X_k^0$ such that $v(y) = v(x) + \phi(v(x))$. It is straightforward to verify that u so constructed is a continuous strictly monotone transformation of v and thus represents $\succsim$ on X^0. Define also $U = u$ on X^0.

Case 1b: $\exists x \in X^0$ such that, $\forall y \in X^0$ we have $\neg(xPy)$. If there exists a v- (equivalently, a $\succsim$-) minimal element in X^0, denote it by x_0 and proceed as in Case 1a. If not, let $\alpha = \sup\left\{ v(x) \,\middle|\, x \in X^0, \nexists y \in X^0, \quad xPy \right\}$ so that $v(x) > \alpha$ implies ($\exists y \in X^0$, xPy) and $v(x) < \alpha$ implies ($\forall y \in X^0$, $\neg(xPy)$). If $\alpha = 0$, in the absence of a minimal element, then we are in Case 1a (where each $x \in X^0$ P-dominates at least one other element). Hence $\alpha > 0$. Define $u(x) = v(x)$ for all x with $v(x) \leq \alpha$ and $\Delta = \alpha$. For x with $v(x) > \alpha$ we repeat the construction above, with X_k^0 including all elements $x \in X^0$ for which the maximal decreasing P-chain is of length k.

Step 2: Definition of u on X^1 and of γ

To extend the function to all of X, partition X^1 into three sets, $X^{1\sim}$ – the elements that have a $\sim$-equivalent in X^0, and $X^{1\prec}$ ($X^{1\succ}$) – those that are worse (better) than all elements in X^0. If $X^{1\sim} \neq \varnothing$, each of $X^{1\prec}$,$X^{1\succ}$ may be empty or not. However, if $X^{1\sim} = \varnothing$ we have to have $X^{1\prec} \neq \varnothing$: otherwise ($X^{1\succ} = X^1$) all $x \in X^1$ and $y \in X^0$ will satisfy $x \succ y$ and

yPy would follow. Further, in this case, since $X^{1\sim} = \varnothing$ and $X^{1\prec} \neq \varnothing$ we also have $X^{1\succ} = \varnothing$, by Lemma 7 (Online Appendix). We will therefore split the definition according to the emptiness of $X^{1\sim}$.

Case 2a: $X^{1\sim} = \varnothing$. In this case we have $X^{1\sim} = X^{1\succ} = \varnothing$ as well as $P = \varnothing$ (as no element in X^1 is ranked as high as any in X^0). Define $u(x) = v(x)\ \forall x \in X^1 = X^{1\prec}$, and set $\gamma = 2(a-b) \geq \Delta$. On X^1, $U(x) = v(x) - \gamma$. Thus $u = v$ is a continuous function on all of X, U represents $\succsim$ on X^0 as well as on X^1, and it also satisfies $U(x) < U(y)$ for every $x \in X^1$ and every $y \in X^0$.

Case 2b: $X^{1\sim} \neq \varnothing$. We first define U that would represent $\succsim$ on the entire space, and then find the $\gamma > 0$ such that $u(x) = U(x) + \gamma \mathbf{1}_{\{x \in X^1\}}$ is continuous. For $x \in X^{1\sim}$, let $y \in X^0$ be such that $x \sim y$ and define $U(x) = U(y)$. This function represents $\succsim$ on $X^0 \cup X^{1\sim}$. We wish to show that it is continuous on $X^{1\sim}$.

Claim: $U : X^0 \cup X^{1\sim} \to \mathbb{R}$ is continuous (also) on $X^{1\sim}$.
Proof: Let there be $(x_n) \to x$ in $X^{1\sim}$ and select corresponding $(y_n), y$ in X^0 (so that $x \sim y$ and $x_n \sim y_n$). Assume first that $x_1 \succ x$ and that $x_1 \succsim x_n \succsim x\ \forall n$. We claim that $u(x_n) \to u(x)$. A symmetric argument would apply to the case $x_1 \prec x$ and ($x_1 \precsim x_n \precsim x$ $\forall n$) and the combination of the two would complete the proof. We thus have $x_1 \sim y_1 \succsim x_n \sim y_n \succsim x \sim y\ \forall n$. By Lemma 7 $\exists \alpha_n \in [0,1]$ such that $\hat{y}_n \equiv \alpha_n y_1 + (1-\alpha_n) y \sim y_n$. By convexity of X^0, $\hat{y}_n \in X^0$. Thus, $U(x_n) = U(y_n) = U(\hat{y}_n)$ and $U(x) = U(y)$. Select a convergent subsequence $(n_k)_k$ such that $\hat{y}_{n_k} \to y^* \in X^0$. As U is continuous on X^0, we have $U(\hat{y}_{n_k}) \to U(y^*)$. Because $x_{n_k} \to x$ are in $X^{1\sim}$ and $\hat{y}_{n_k} \to y^*$ are in X^0, the two sequences are comparable and A2 implies that $x \sim y^*$ and thus also $y \sim y^*$. It follows that $U(x) = U(y) = U(y^*) = \lim U(\hat{y}_{n_k}) = \lim U(x_{n_k})$. □

Let $v_* = \inf_{x \in X^{1\sim}} v(x)$ and $v^* = \sup_{x \in X^{1\sim}} v(x)$. Recall that v is bounded (by b, a) and thus $b \leq v_* \leq v^* \leq a$. Denote $u^* = \sup_{x \in X^{1\sim}} U(x)$ and $u_* = \inf_{x \in X^{1\sim}} U(x)$ (which can be ∞, $-\infty$, respectively).

On $X^{1\sim}$, both v and U represent $\succsim$, and are continuous. Thus there exists a continuous, strictly increasing $\psi : (v_*, v^*) \to (u_*, u^*)$ such that, $\forall x \in X^{1\sim}$, $U(x) = \psi(v(x))$ and $\lim_{v \searrow v_*} \psi(v) = u_* \quad \lim_{v \nearrow v^*} \psi(v) = u^*$. Further, if v_* is obtained by v on $X^{1\sim}$, $u_* > -\infty$ and we can define $\psi(v_*) = u_*$, and, similarly, $\psi(v^*) = u^*$ in case $v^* = \max_{x \in X^{1\sim}} v(x)$ (and $u^* < \infty$).

Next, extend ψ to the entire range of v on X^1. Consider first $v > v^*$. If $X^{1\succ} = \varnothing$, then v on X^1 is bounded above by v^*, and the extension of ψ to this range is immaterial.

Otherwise, that is, $X^{1\succ} \neq \emptyset$, A4 implies $U(x) < \infty\ \forall x \in X^0$ and hence $u^* < \infty$. Set $\psi(v) = (v - v^*) + u^*\ \forall v > v^*$, representing $\succsim$ on $X^{1\succ}$. Similarly, consider $v < v_*$. If $X^{1\prec} = \emptyset$, then v on X^1 is bounded below by v_*, and the extension of ψ to this range is immaterial. Otherwise, that is, $X^{1\prec} \neq \emptyset$, we know, by A4, that $U(x) > -\infty$ for all $x \in X^0$ and this means that $u^* > -\infty$. Hence we can set $\psi(v) = (v - v_*) + u_*$ for all $v < v_*$. Thus, $U(x) = \psi(v(x))$ is well defined for all $x \in X^1$; combined with the definition of U on X^0, we know that (i) U represents $\succsim$ on the entire space X; (ii) U is continuous on each of X^0 and X^1. It remains to define $\gamma > 0$ and show that, for that γ, $u(x) = U(x) + \gamma \mathbf{1}_{\{x \in X^1\}}$ is continuous on the entire space. We set γ to be equal to Δ as defined in Step 1.

Claim: $u : X \to \mathbb{R}$ is continuous on X.

Proof: We only need to consider sequences $(x_n) \subset X^1$ that converge to $x \in X^0$. Let there be given such a sequence $(x_n) \subset X^1$ with $x_n \to x \in X^0$. Distinguish between two cases:

Case 2b(i): $\exists y \in X^0$, xPy. Assume w.l.o.g. that $u(y) = u(x) - \gamma$, i.e., that y is a u-maximal element with xPy. There exists a sequence $(x'_n) \subset X^1$ with $x'_n \to x \in X^0$ and $x'_n \sim y$ so that $U(x'_n) = U(y) = u(y)$ and, U on X^1 being a continuous transformation of v, where the latter is continuous on all of X, we also have $U(x_n) \to \lim_n U(x'_n) = u(y) = u(x) - \gamma = U(x) - \gamma$. Hence $u(x_n) = U(x_n) + \gamma \to U(x) = u(x)$ as required.

Case 2b(ii): $\nexists\, y \in X^0$ with xPy. By the definition of $u = U$ on X^0 in Step 1, we are in Case 1b and $u(x) = U(x) = v(x)$. Consider the sequence (x_n). Because it is convergent, and v is continuous on X, $\exists \lim_n v(x_n)$ $(= v(x))$. On X^1 $U(\cdot) = \psi(v(\cdot))$ is continuous, thus $\exists \lim_n U(x_n)$. The limit $v(x) = \lim_n v(x_n)$ cannot exceed v_* (if it did, $\exists y \in X^0$ such that $x_n \succsim y$ for infinitely many n's, and xPy would follow). However, in the domain $v \leq v_*$ we have $\psi(v) = (v - v_*) + u_* = v - (v_* - u_*)$. Further, in this case (corresponding to Case 1b in the definition of u on X^0), $u_* = \inf_{x \in X^0} v(x) = 0$ while $v_* = \inf_{x \in X^{1\sim}} v(x) = \Delta = \gamma$. It follows that

$$\lim_n U(x_n) = \lim_n \psi(v(x_n)) = \lim_n v(x_n) - \gamma = v(x) - \gamma$$

and thus $u(x_n) = U(x_n) + \gamma \to v(x) = u(x)$ and continuity is established. ■

B.2 A Lipschitz Continuity Property

This section shows that A4 can be strengthened to the following Lipschitz continuity property.

A6 Lipschitz: There exists $\delta > 0$ such that, for every $x, y, z \in X^0$, and every sequence

$z_n \to z$ with $(z_n) \subset X^1$ such that $x \gtrsim z$ and $z_n \gtrsim y$, we have $\|x - y\| > \delta$.

Axiom A6 states that, for a bundle $x \in X^0$ to be better than another bundle, $y \in X^0$, by "at least the cost of the principle", x should not be too close to y. We dub it "Lipschitz" as it will be satisfied by any utility function that is Lipschitz continuous on the entire space. Observe, however, that we only require the Lipschitz condition for one specific $\delta > 0$, guaranteeing that two bundles that are δ-close will not have a utility gap that is higher than a certain threshold (the presumed γ). If we restrict attention to compact bundle spaces, we can use A6 in lieu of A4:

Corollary 1 *Let $d \in \{0,1\}^n$ be given and assume that X is compact. If the relation $\gtrsim$ on X satisfies A1-A3, A5, and A6, there exist a continuous function $u : X \to \mathbb{R}$, which isn't constant on X^0, and a constant $\gamma > 0$ such that $\gtrsim$ is represented by $U(x) = u(x) - \gamma \mathbf{1}_{\{d \cdot x > 0\}}$*

Proof: First, it useful to re-state axiom A6 in terms of the relation P as follows:

There exists $\delta > 0$ such that, for every $x, y \in X^0$, if xPy then $\|x - y\| > \delta$.

We next show that, when X is compact, A6 implies A4. Since X is compact, we have that X^0 is compact as well. We wish to show that no infinite decreasing P chain can be bounded from below, nor can an infinite increasing P chain be bounded from above. However, A6 would make a stronger claim, namely, that there are no infinite P chains (neither increasing nor decreasing). Indeed, Lemma 2 implies that P is transitive. Had there been an infinite P chain, we would have to find two elements, say x_i and x_j such that $x_i P x_j$ (with $i > j$ for a decreasing P chain and $i < j$ for an increasing one) while they are in a δ-neighborhood of each other, in contradiction to A6. $\square$

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Consumption of Values

Online Appendix

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This online appendix is organized in two subsections. Subsection C.1 contains an auxiliary result. Subsection C.2 presents some examples showing that all axioms presented in the paper are independent of each other.

C.1 An Auxiliary Result

This appendix presents and proves the following auxiliary result used in the proof of Theorem 1.

Theorem 2 *Let* $\succsim$ *on* X *satisfy A1-A3. Then, a bounded and continuous function* $u : X^1 \to \mathbb{R}$ *that represents* $\succsim$ *on* X^1 *has a unique continuous extension to (all of)* X*. This extension represents* $\succsim$ *also on* X^0*.*

Note that the theorem does not state that the extended u represents $\succsim$ on X in its entirety. Indeed, the continuity axioms do not state that preferences change continuously along a sequence that crosses from X^1 to X^0, and thus a utility function that is continuous on the entire space cannot be expected to represent preferences across the two subspaces. As mentioned earlier, Kopylov (2016, Theorem 6) provides a related extension result by imposing a property dubbed "Cauchy continuity" on X^1. This condition is weaker than our A3 axiom. Indeed, in order to show that there is an extension of a continuous u from X^1 to X^0, one can make do with Cauchy continuity and invoke Kopylov's result. However,

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in our setup the relation $\succsim$ is defined also on X^0, and in order to verify that the extension represents $\succsim$ also on X^0, an additional assumption is needed. We use A2, and later show that it is independent of A3, and therefore isn't implied by Cauchy continuity of $\succsim$ on X^1.

Proof of Theorem 2. Without loss of generality we assume that d isn't identically 0 not identically 1, so that $X^0, X^1 \neq \varnothing$. Note that, due to convexity of X^1, X^0 is included in the closure of X^1.

We start with a few lemmas. Throughout we assume that $\succsim$ on X satisfies A1-A3. (Note, however, that the first three lemmas do not make use of A3).

Lemma 6 Let there be a sequence $x_n \to x$. Assume that $[(x_n) \subset X^0$ and $x \in X^0]$ or $[(x_n) \subset X^1$ and $x \in X^1]$. Then, for all $y \in X$, if $x_n \succsim y$, then $x \succsim y$ and if $y \succsim x_n$, then $y \succsim x$.

Proof: Define $y_n = y$ for all $n \geq 1$. Note that the sequences $x_n \to x$ and $y_n \to y$ are comparable (satisfying Condition A), and apply A2. $\square$

Lemma 7 Let there be $x, y, z \in X$ with $x \succ y \succ z$. Assume that $x, z \in X^0$ or that $x, z \in X^1$. Then there exists $\alpha \in [0,1]$ such that $y \sim \alpha x + (1-\alpha) z$.

Proof: The argument is familiar, and we mention it explicitly to point out that it does not depend on monotonicity or openness conditions. Let there be $x, y, z \in X$ with $x \succ y \succ z$ and assume without loss of generality that $x, z \in X^0$ (the argument is identical for X^1). Define

$$\begin{aligned} A^- &= \{\alpha \in [0,1] \mid y \succ \alpha x + (1-\alpha) z\} \\ A^+ &= \{\alpha \in [0,1] \mid \alpha x + (1-\alpha) z \succ y\} \end{aligned}$$

and we have $A^- \cap A^+ = \varnothing$, with $1 \in A^+$ and $0 \in A^-$. Consider $\alpha^* = \inf A^+$ and define $x^* = \alpha^* x + (1-\alpha^*) z$. We wish to show that it is the desired α, so that $\alpha^* \notin A^- \cup A^+$ and $y \sim x^*$ holds. Suppose that this is not the case. If $\alpha^* \in A^-$ (and $y \succ x^*$), we can choose a sequence $\alpha_n^+ \in A^+$ with $\alpha_n^+ \searrow \alpha^*$. Then $x_n = \alpha_n^+ x + (1-\alpha_n^+) z \in A^+ \to x^*$. Importantly, X^0 is convex. Hence $x_n \in X^0$ for all n and $x^* \in X^0$ as well. Lemma 6 implies that $x^* \succsim y$, a contradiction. Similarly, if $\alpha^* \in A^+$ (and $x^* \succ y$), then $\alpha^* = \min A^+$ and we must have $\alpha^* > 0$ as $0 \in A^-$, in which case we can choose a sequence $\alpha_n^- \in A^-$ with $\alpha_n^- \nearrow \alpha^*$. Then, Lemma 6 implies that $y \succsim x^*$, again a contradiction. Hence $y \sim x^*$.

The argument holds also for X^1 since it is a convex set as well. $\square$

We also note the following.

Lemma 8 For all comparable sequences $\xi_n \to \xi$ and $\eta_n \to \eta$, if $\xi \succ \eta$, then there exists an $N > 0$ such that

$$\xi_n \succ \eta_m \quad \forall n, m > N.$$

Proof: If the conclusion does not hold, for $N_1 = 1$ we have n_1, m_1 such that $\eta_{m_1} \succcurlyeq \xi_{n_1}$. Set $N_2 = \max(n_1, m_1)$ and find $n_2, m_2 > N_2$ such that $\eta_{m_2} \succcurlyeq \xi_{n_2}$. Continuing this way, we generate two subsequences $(n_k, m_k)_k$ such that $\eta_{m_k} \succcurlyeq \xi_{n_k}$ for all k, with $\xi_{n_k} \to \xi$ and $\eta_{m_k} \to \eta$ being comparable (as subsequences of comparable sequences with these limits). A2 would then imply $\eta \succcurlyeq \xi$, a contradiction. $\square$

Two implications of the A3 (in the presence of A1, A2) will be useful to state explicitly.

Lemma 9 For all comparable sequences $x_n \to x$ and $y_n \to x$, and all $z, w \in X$, if ($x_n \succsim z$ and $w \succsim y_n$) then $w \succsim z$.

Proof: Let there be given comparable sequences $x_n \to x$ and $y_n \to x$ as well as $z, w \in X$ such that $x_n \succsim z$ and $w \succsim y_n$. We need to show that $w \succsim z$. Assume, to the contrary, that $z \succ w$. Define $y = x$. With $x_n \succsim z \succ w \succsim y_n$ we can apply A3 and conclude that $x \succ y$ which is impossible as $y = x$. Thus we rule out the possibility $z \succ w$ and conclude that $w \succsim z$ as required. $\square$

The following lemma is not needed for Theorem 2 but it is used in the proof of Theorem 1. It is similar to A3 and can easily be shown to imply it. Thus the lemma shows that, in the presence of A1 and A2, the two conditions are equivalent.

Lemma 10 For all pairs of comparable sequences, ($x_n \to x$ and $y_n \to y$) and ($z_n \to z$ and $w_n \to w$), if (i) $z \succ w$; and (ii) $x_n \succsim z_n$; $w_n \succsim y_n$ for all n, then $x \succ y$.

Proof: Assume, then, that ($x_n \to x$ and $y_n \to y$) and ($z_n \to z$ and $w_n \to w$), are given, such that (i) $z \succ w$; and (ii) $x_n \succsim z_n$; $w_n \succsim y_n$ for all n. We split the argument depending on the reason that $z_n \to z$ and $w_n \to w$ are comparable. Assume, first, that they satisfy Condition A, that is, that $(z_n) \subset X^i, z \in X^i$ and $(w_n) \subset X^j, w \in X^j$ for $i, j \in \{0, 1\}$. In this case, because the limit of each sequence $(z_n), (w_n)$ belongs to the same space X^i as the sequence itself, we also have, w.l.o.g., $z_n \succ w$ and $z \succ w_n$ for all n. (Otherwise, we can apply A2 to the relevant sequence and to a constant sequence and derive $w \succsim z$ from A2.) Next, consider a specific $n > N$. If there are infinitely many indices $n_k > n$ such that $z_{n_k} \succsim z_n$, let n be the minimal index with this property, and, for that n, set $z^* = z_n$ and restrict attention to the subsequence $(n_k)_k$. Clearly, $x_{n_k} \succsim z_{n_k} \succsim z_n = z^*$. If not, then for every $n > N$ there is $l_n > 0$ such that, for all $m > n + l_n$, we have $z_n \succ z_m$. In that case we can select a subsequence (z_{n_k}) such that $z_{n_k} \succ z_{n_{k+1}}$. As $(z_{n_k}) \to z$ and belongs to the same space (as z), we can compare it to the sequence that equals z throughout and conclude that $z_{n_k} \succsim z$ for all k. We can then set $z^* = z$ and we have $x_{n_k} \succsim z_{n_k} \succsim z = z^*$. Thus we found an element z^* and a subsequence (n_k) such that $x_{n_k} \succsim z^*$ with z^* being either z or one of z_n.

We now limit attention to the subsequence (n_k) and repeat the argument for (w_n). In a symmetric fashion, we now have a sub-subsequence (n_{k_l}) and w^* which is either w or one of w_{n_k} such that $w^* \succsim w_{n_{k_l}} \succsim y_{n_{k_l}}$. Importantly, whether $z^* = z_n$ or $z^* = z$, whether $w^* = w_{n_k}$ or $w^* = w$, we have $z^* \succ w^*$ (where this follows either from $z \succ w$, which was given, or from the claims proven above for the other three possibilities). Thus A3 can be used to derive the conclusion $x \succ y$.

Next assume that $z_n \to z$ and $w_n \to w$ are comparable but that they do not satisfy Condition A. This means that they satisfy Condition B, that is, that $(z_n) \subset X^i, z \in X^j$ and $(w_n) \subset X^i, w \in X^j$ for $i, j \in \{0, 1\}$. But this also means that $i \neq j$ (or else Condition A would also hold). Further, because X^0 is closed, we have to have $(z_n), (w_n) \subset X^1$ while $z, w \in X^0$. As X^0 is convex, hence connected, we have $z' \in X^0$ such that $z \succ z' \succ w$ (otherwise, we could use Lemma 6, applied to $z_n \to z$ and $w_n = w$ to get $w \succsim z$). Repeating the argument for the pair $z' \succ w$, we conclude that there is also $w' \in X^0$ such that

$$z \succ z' \succ w' \succ w.$$

Next we select elements $(z'_n), (w'_n) \subset X^1$ such that $z'_n \to z'$ and $w'_n \to w'$. Notice that this is possible as X^0 is a non-trivial subspace of X. Thus we have four sequences, $z_n \to z$, $w_n = w$, $z'_n \to z'$, $w'_n \to w'$ and two of which are comparable. Applying Lemma 8 consecutively, we conclude that there exists an $N > 1$ such that, for all $n, k, l, m > N$ we have

$$z_n \succ z'_k \succ w'_l \succ w_m.$$

Fix $k, l > N$ and set $z^* = z'_k$, $w^* = w'_l$. Thus, $z_n \succ z^* \succ w^* \succ w_n$ for all $n > N$. As we also have $x_n \succcurlyeq z_n$ and $w_n \succcurlyeq y_n$ for all n, we conclude that $x_n \succ z^* \succ w^* \succ y_n$ and apply A3 to conclude that $x \succ y$. $\square$

We now turn to define the extension. Let there be given a bounded and continuous function $u : X^1 \to \mathbb{R}$ that represents $\succsim$ on X^1. We first note that

Lemma 11 Assume that $(x_n) \subset X^1$ is such that $x_n \to y \in X^0$. Then $\exists \lim_{n\to\infty} u(x_n)$.

Proof: Assume that $x_n \to y \in X^0$. We claim that there exists $a \in \mathbb{R}$ such that $u(x_n) \to a$. If $u(x_n) \to \sup_{x\in X^1} u(x)$ or $u(x_n) \to \inf_{x\in X^1} u(x)$ then $u(x_n)$ is convergent and we are done. Assume, then, that this is not the case. As u is bounded, we can find a number $a \in (\inf_{x\in X^1} u(x), \sup_{x\in X^1} u(x))$ and a subsequence $(x_{n_k})_k$ such that $u(x_{n_k}) \to_{k\to\infty} a$. If we also have $u(x_n) \to_{n\to\infty} a$, we are done. Otherwise, there exists $\varepsilon > 0$ such that, for infinitely many n's, $u(x_n) > a + \varepsilon$, or that, for infinitely many n's, $u(x_n) < a - \varepsilon$ (or both). This means that there is another subsequence $(x_{n_l})_l$ such that $u(x_{n_l}) \to_{l\to\infty} b$ with $|a - b| \geq \varepsilon$. Assume w.l.o.g. that $b \geq a + \varepsilon$. As u is continuous on X^1, and the latter is convex (and connected), we have points $z, w \in X^1$ such that $b - \frac{\varepsilon}{3} > u(z) > u(w) > a + \frac{\varepsilon}{3}$. But this means that, for large enough k, l, we have $x_{n_l} \succ z \succ w \succ x_{n_k}$ with $x_{n_k} \to_{k\to\infty} y$ and $x_{n_l} \to_{l\to\infty} y$. By A3 we should get $y \succ y$, a contradiction. Thus $u(x_n)$ is convergent. $\square$

Lemma 12 For every $y \in X^0$ there exists $a \in \mathbb{R}$ such that, for every $(x_n) \subset X^1$ with $x_n \to y$, we have $\exists \lim_{n\to\infty} u(x_n) = a$.

Proof: Lemma 11 already established that every convergent sequence $x_n \to y \in X^0$ generates a convergent sequence of utilities. Clearly, this means that the limit is independent of the sequence. Explicitly, if $(x_n), (x_n') \subset X^1$ are such that $x_n \to y \in X^0$ and $x_n' \to y$, we know that for some $a, a' \in \mathbb{R}$ we have $u(x_n) \to a$ and $u(x_n') \to a'$. But if $a \neq a'$, we can generate a combined sequence whose utility has no limit. (Say, for $z_{2n} = x_n, z_{2n+1} = x_n'$, we get $z_n \to y$ but $u(z_n)$ is not convergent.) $\square$

We can finally define the extension of u. For every $y \in X^0$ there exist sequences $(x_n) \subset X^1$ with $x_n \to y$. By Lemma 11 we have $\exists \lim_{n\to\infty} u(x_n)$ and by Lemma 12 its value is independent of the choice of the convergent sequence. Thus, setting

$$u(y) = \lim_{n\to\infty} u(x_n)$$

is well-defined. Observe that this is the unique extension of u to X^0 that holds a promise of continuity.

Lemma 13 u is continuous (also) on X^0.

Proof: Let there be given $y \in X^0$ and a convergent sequence $x_n \to y$. We need to show that $u(x_n) \to u(y)$. We will consider two special cases: $(x_n) \subset X^1$ and $(x_n) \subset X^0$. If we show that for each of these the conclusion $u(x_n) \to u(y)$ holds, we are done, as any other sequence can be split into two subsequences, one in X^0 and the other in X^1, and each of these, if infinite, has to yield u values that converge to $u(y)$.

When we consider $(x_n) \subset X^1$ we are back to the first part of the proof, where we showed that $u(x_n)$ is convergent, and that its limit has to be $u(y)$. Consider then a sequence $(x_n) \subset X^0$ such that $x_n \to y$ and assume that $u(x_n) \to u(y)$ doesn't hold. Then there exists $\varepsilon > 0$ such that, for infinitely many n's, $u(x_n) > u(y) + \varepsilon$, or that, for infinitely many n's, $u(x_n) < u(y) - \varepsilon$ (or both). For each n select a sequence $\left(x_n^k\right)_k \subset X^1$ such that $x_n^k \to_{k\to\infty} x_n$. For every m, pick n such that $\|x_n - y\| < \frac{1}{2m}$ and k such that $\left\|x_n^k - x_n\right\| < \frac{1}{2m}$ so that $(x_n^n) \subset X^1$ and $x_n^n \to_{n\to\infty} y$. However, $|u(x_n^n) - u(y)| \geq \varepsilon$, a contradiction. We thus conclude that u is continuous on X^0. $\square$

Next, we wish to show that the continuous extension we constructed represents $\succsim$ also on its extended domain, X^0. We do this in two steps. First, we observe the following:

Lemma 14 For all $x, y \in X^0$, if $u(x) > u(y)$ then $x \succ y$.

Proof: By definition of u, we can take sequences $(x_n), (y_n) \subset X^1$ such that $x_n \to x$ and $y_n \to y$. Letting $\varepsilon = u(x) - u(y) > 0$ choose N large enough so that for all $n \geq N$ we have $|u(x_n) - u(x)|, |u(y_n) - u(y)| < \varepsilon/3$. As u is continuous on X^1 we can also find $z^*, w^* \in X^1$ so that $u(z^*) = u(x) - \varepsilon/3$; $u(w^*) = u(y) + \varepsilon/3$. Thus $u(x_n) > u(z^*) > u(w^*) > u(y_n)$ for all $n \geq N$. A3 implies that $x \succ y$. $\square$

The next and final step of the proof is to show the converse, namely:

Lemma 15 For all $x, y \in X^0$, if $u(x) = u(y)$ then $x \sim y$.

Proof: We first prove an auxiliary claim:

Claim 1 *Assume that, for $z, w \in X^0$, $u(z) = u(w) = a$ but $z \succ w$. Let $(z_n), (w_n) \subset X^1$ converge to z and w respectively. Then $\exists N$ such that, $\forall n \geq N$ we have (i) $u(z_n) \geq a$ and (ii) $u(w_n) \leq a$.*

Proof of Claim: Suppose first that $u(z_n) < a$ occurs infinitely often. Let (n_k) be a sequence such that $u(z_{n_k}) < a$. Because $u(w_n) \to a$, for each such k we can find $m(n_k)$ such that $u\left(w_{m(n_k)}\right) > u(z_{n_k})$ and $m(n_k)$ increases in k. Thus we have two sequences $(z_{n_k}), \left(w_{m(n_k)}\right) \subset X^1$, converging to z and w, respectively, with $w_{m(n_k)} \succ z_{n_k}$. By A2, we get $w \gtrsim z$, a contradiction. By a similar argument, if $u(w_n) > a$ occurs infinitely often, we select such a subsequence $u(w_{n_k}) > a$ and $u\left(z_{m(n_k)}\right) < u(w_{n_k})$ and $w \gtrsim z$ follows again. Thus, $\exists N$ such that, $\forall n \geq N$ we have both $u(z_n) \geq a$ and $u(w_n) \leq a$. □

Equipped with this Claim we turn to prove the lemma. Assume that $x, y \in X^0$ satisfy $u(x) = u(y)$ but $x \succ y$. Because X^0 is connected and $\gtrsim$ satisfies A2, we have to have $z \in X^0$ such that $x \succ z \succ y$. Applying the same reasoning to z and y we can also get $w \in X^0$ such that $x \succ z \succ w \succ y$.

Let $a = u(x) = u(y)$. Applying Lemma 14, we know that $x \succ z \succ w \succ y$ and, indeed, $x \gtrsim z \gtrsim w \gtrsim y$ implies $u(x) \geq u(z) \geq u(w) \geq u(y)$ and thus we have $u(x) = u(z) = u(w) = u(y) = a$.

Let there be sequences $(x_n), (z_n), (w_n), (y_n) \subset X^1$ converging to x, z, w, y, respectively. Applying the Claim to $x \succ z$, we conclude that, from some N_1 on, $u(z_n) \leq a$. Applying the same Claim to $w \succ y$, we find that, from some N_2 on, $u(w_n) \geq a$. However, when we apply it to $z \succ w$ we find that, from some N_3 on, $u(z_n) \geq a$ and $u(w_n) \leq a$. For $n \geq \max(N_1, N_2, N_3)$ we have $u(z_n) = u(w_n) = a$. This means that $z_n \sim w_n$ and A2 yields $z \sim w$, a contradiction. □ ■

C.2 Examples

We use two continuity axioms, A2 and A3. A2 seems to be rather strong, and, as mentioned above, if we drop the comparability restriction, it is, per se,[1] stronger than the standard continuity assumption of consumer theory. Moreover, if we drop the comparability restriction, the two axioms are equivalent (for a weak order). Specifically, if we define

A2*. Universal Weak Preference Continuity: For all sequences $x_n \to x$ and $y_n \to y$, if $x_n \gtrsim y_n$ for all n, then $x \gtrsim y$.

A3*. Universal Strict Preference Continuity: For all sequences $x_n \to x$ and $y_n \to y$, and all $z, w \in X$, if $x_n \gtrsim z \succ w \gtrsim y_n$ for all n, then $x \succ y$.

[1] That is, without A1 necessarily assumed.

We can state

Observation 1 *If $\succsim$ is a weak order on X, then A2* and A3* are equivalent.*

Proof: Assume first that $\succsim$ satisfies A2*. Then for the bundles in A3* we have $x \succsim z$ and $w \succsim y$, which implies $x \succ y$ by transitivity.

Next, assume that $\succsim$ satisfies A3*. We first claim that, for all sequences $x_n \to x$ and $y_n \to y$, if $x \succ y$, then there exists an N such that $x \succ y_n$ and $x_n \succ y$ for all $n > N$. To see this, suppose that the contrary holds. If $y_n \succsim x$ for infinitely many n's, then for these n's we have $y_n \succsim x \succ y \succsim y$, which by A3* implies $y \succ y$, a contradiction. Alternatively, $y \succsim x_n$ for infinitely many n's would imply $x \succsim x \succ y \succsim x_n$ and $x \succ x$.

To see that A2* holds, let there be given sequences $x_n \to x$ and $y_n \to y$, such that $x_n \succsim y_n$ for all n, and assume that, contrary to our claim, $y \succ x$. For all n large enough, $y \succ x_n \succsim y_n \succ x$ by the argument above. Fix such a k so that $y \succ x_k \succsim y_k \succ x$. Apply the argument again to conclude that, for some N, we have $y_n \succ y_k$ for all $n > N$. Since $x_n \succsim y_n$ for all n, we have by transitivity $x_n \succ y_k$ for all $n > N$. So we have $x_n \succ y_k \succ x \succsim x$ for all $n > N$, which by A3* implies $x \succ x$, an impossibility. □

In light of this equivalence of the "universal" versions of the axioms (applying to all sequences, rather than only to comparable ones), one may wonder whether A3 is also needed, and, if so, maybe A3 can be assumed but A2 can be dispensed with. In the following we provide a few examples that show that none of the axioms is redundant. In the first five examples we have $n = 2$, $X = [0,10]^2$ and $d = (1,0)$, so that the principle is satisfied on the x_2 axis (X^0 consists of all the points with $x_1 = 0$) but not off the axis (X^0 consists of all the points with $x_1 > 0$). We define $\succsim$ by a numerical function v so that A1 is satisfied in all examples.

C.2.1 Example 1: A2 without A3 (I)

Let v be given by[2]:

$$v(x_1,x_2) = \begin{cases} 3 & x_1 = 0 \\ \sin\left(\frac{1}{x_1}\right) & x_1 > 0 \end{cases}$$

So the x_2 axis ($x_1 = 0$) is an indifference class that is preferred to anything else. Preference off the axis depend only on x_1, in a continuous way on the interior ($x_1 > 0$), but in a way

[2]Here and in the sequel we drop one set of parentheses for clarity. That is, $u_d((x_1,x_2))$ is denoted $u_d(x_1,x_2)$.

that has no limit as we approach $x_1 = 0$.

To see that A2 is satisfied, consider $x_n \to x$ and $y_n \to y$ with $x_n \succsim y_n$ as in the antecedents of A2. Then if $x, y \in X^0$, the consequent $x \succsim y$ follows as $x \sim y$ for any $x, y \in X^0$. And if $x, y \in X^1$, then from some point on $x_n, y_n \in X^1$ and the consequent follows from the continuity of v on X^1. However, A3 isn't satisfied. More specifically, the claim of Lemma 9, which is an implication of A3, does not hold. To see this, define $x_n = \left(\frac{1}{\left(2n+\frac{1}{2}\right)\pi}, 1\right)$; $y_n = \left(\frac{1}{\left(2n+\frac{3}{2}\right)\pi}, 1\right)$ and $x = (0,1)$ so that $x_n, y_n \to x$. Let $z = \left(\frac{2}{\pi}, 1\right)$ and $w = \left(\frac{2}{3\pi}, 1\right)$ so that $v(x_n) = v(z) = 1$ and $v(y_n) = v(w) = -1$. Thus, $x_n \succsim z$ and $w \succsim y_n$ but $w \succsim z$ doesn't hold. $\square$

C.2.2 Example 2: A2 without A3 (II)

The previous example relies on the absence of a limit – preferences on X^1 have no "Cauchy sequences". The next example shows that this is only one problem that may arise, and that A3 may not hold even if preferences are very well-behaved on each of X^0, X^1. Let v be given by:

$$v(x_1, x_2) = \begin{cases} x_2 & x_1 = 0 \\ x_2 - 3 & x_1 > 0, x_2 < 5 \\ x_2 - 2 & x_1 > 0, x_2 = 5 \\ x_2 - 1 & x_1 > 0, x_2 > 5 \end{cases}$$

In the subspace $x_1 > 0$, $\succsim$ could also be represented by $v'(x_1, x_2) = x_2 - 2$ and it is clearly continuous there. But v is defined by taking $v'(x_1, x_2) = 3$ (corresponding to $x_2 = 5$) as a watershed, shifting the region $v'(x_1, x_2) > 3$ (corresponding to $x_2 > 5$) up by 1 and the region $v'(x_1, x_2) < 3$ (corresponding to $x_2 < 5$) down by 1. This generates "holes" in the range of U that could be skipped if we only had to worry about $x_1 > 0$. Yet, we cannot re-define U on this range to be continuous because we have points on the x_2 axis ($x_1 = 0$) that are in between preference-wise.

To see that A2 is satisfied, consider $x_n \to x$ and $y_n \to y$ with $x_n \succsim y_n$ as in the antecedents of A2. Then if $x, y \in X^0$, the consequent $x \succsim y$ follows because v is obviously continuous on X^0. And if $x, y \in X^1$, then from some point on $x_n, y_n \in X^1$ and the consequent follows from the fact that on X^1 the relation $\succsim$ could also be represented by v' which is continuous on X^1. However, A3 is violated. To see this, let $x_n = \left(1, 5 + \frac{1}{n}\right)$ and $y_n = \left(1, 5 - \frac{1}{n}\right)$ with $x = (1,5)$ being their common limit. Take $z = (0,4)$ and $w = (0,3)$ so that $x_n \succsim z$ and

$w \succsim y_n$ because $v\left(1,5+\frac{1}{n}\right) = 4+\frac{1}{n} > v(0,4)$ and $v(0,3) = 3 > 2+\frac{1}{n} = v\left(1,5-\frac{1}{n}\right)$. However, $w \succsim z$ doesn't hold. Thus, the claim of Lemma 9 is again violated. □

C.2.3 Example 3: Lemma 9

The next example satisfies the conclusion of Lemma 9 but not the other properties. Let v be defined by:

$$v(x_1, x_2) = \begin{cases} x_2 & x_1 = 0 \\ x_2 - 1 & x_1 > 0, x_2 < 5 \\ 9 - x_2 & x_1 > 0, x_2 \geq 5 \end{cases}$$

That is, as long as $x_2 \leq 5$ preferences are monotone in x_2 with a "jump" at the x_2 axis. However, when x_2 is above 5, the direction of preferences in the interior $(x_1 > 0)$ reverses, but not on the axis.

These preferences do not satisfy A2. For example, let $x_n = \left(\frac{1}{n}, 4\right), y_n = \left(\frac{1}{n}, 6\right)$ with $x = (0,4)$ and $y = (0,6)$. Then we have $v(x_n) = v(y_n) = 3$ and thus $x_n \succsim y_n$, but $v(x) = 4 < 6 = v(y)$ so that $x \succsim y$ fails to hold.

At the same time, the conclusion of Lemma 9 holds. To see this, let $x_n \to x$ and $y_n \to x$. As v is uniformly continuous both on X^0 and on X^1, $\lim v(x_n)$ and $\lim v(y_n)$ exist and they are equal. This means that there can be no $a = v(z)$ and $b = v(w)$ such that $v(x_n) \geq a > b \geq v(y_n)$ for all n, and if $x_n \succsim z$ and $w \succsim y_n$ for all n, $w \succsim z$ has to follow.

Finally, these preferences also do not satisfy A3. To see this, we can take $x_n = \left(\frac{1}{n}, 4\right), y_n = \left(\frac{1}{n}, 7\right)$ so that $v(x_n) = 3$ and $v(y_n) = 2$. For $z = (0,3)$ and $w = (0,2)$ we have $v(z) = 3, v(w) = 2$ so that $x_n \succsim z \succ w \succsim y_n$. But the limit points, $x = (0,4)$ and $y = (0,7)$ do not satisfy $x \succ y$ (in fact, the converse holds, that is, $y \succ x$). □

C.2.4 Example 4: A2 and Lemma 9 without A3

Next consider v defined by :

$$v(x_1, x_2) = \begin{cases} x_2 - 2 & x_1 > 0 \\ x_2 & x_1 = 0, \quad x_2 < 4 \\ 4 & x_1 = 0, \quad 4 \leq x_2 \leq 5 \\ x_2 - 1 & x_1 = 0, \quad x_2 > 5 \end{cases}$$

Thus, along the axis $x_1 = 0$, preferences are represented by a non-decreasing continuous function of x_2 that is constant on a given interval, and off it ($x_1 > 0$) they could also be represented by x_2.

We claim that these preferences satisfy A2 and the conclusion of Lemma 9 but not A3. Starting with A2, consider $x_n \to x$ and $y_n \to y$ with $x_n \succsim y_n$ as in the antecedents of A2. Then if $x_n, y_n \in X^0$, the consequent $x \succsim y$ follows because v is continuous on X^0. And if $x_n, y_n, x, y \in X^1$, the consequent follows from the fact that on X^1 the relation $\succsim$ could also be represented by $v' = x_2$. We are left with the interesting case in which $x_n, y_n \in X^1$ but $x, y \in X^0$. Because $x_n \succsim y_n$, we know that the second component of x_n is at least as high as is that of y_n, and it follows that the same inequality holds in the limit and $x \succsim y$.

The conclusion of Lemma 9 also holds because v is uniformly continuous on each of X^0 and X^1. Thus, $x_n \to x$ and $y_n \to x$ imply that $\lim v(x_n) = \lim v(y_n)$ (and that both exist).

However, A3 fails to hold. To see this, consider $x_n = \left(\frac{1}{n}, 4\right), y_n = \left(\frac{1}{n}, 5\right)$ with $x = (0, 4)$ and $y = (0, 5)$. For $z = (0, 3)$ and $w = (0, 2)$ we have $U(z) = 3, U(w) = 2$ so that $y_n \succsim z \succ w \succsim x_n$. But for limit points $x \sim y$, in violation of the axiom. □

C.2.5 Example 5: A3 without A2

Finally, we show that A3 does not imply A2. Let

$$v(x_1, x_2) = \begin{cases} -1 & x_1 > 0 \\ x_2 & x_1 = 0 \end{cases}$$

That is, the entire X^1 is a single indifference class that is below, preference-wise, the entire x_2 axis. We claim that these preferences satisfy A3 but not A2.

To see that A3 holds, consider $(x_n), (y_n)$ and x, y, z, w in X such that $x_n \to x$ and $y_n \to y$ and $x_n \succsim z \succ w \succsim y_n$. If $(x_n), (y_n) \subset X^0$ then we have $x, y \in X^0$. Because v is simply x_2 on X^0, the conclusion follows. If $(x_n), (y_n) \subset X^1$ we cannot have $x_n \succsim z \succ w \succsim y_n$ because $x_n \sim y_n$. Thus, A3 holds. However, A2 can easily seen to be violated. For example, $x_n = \left(\frac{1}{n}, 4\right), y_n = \left(\frac{1}{n}, 5\right)$ satisfy $x_n \succsim y_n$ but at the limit we get $(0, 5) \succ (0, 4)$. □

C.2.6 Example 6: The Role of Connectedness

The following example shows that for Theorem 2 to hold, the set X has to be connected. Let

$$X = \left\{ (x_1, x_2) \;\middle|\; \begin{array}{c} 0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \\ or \\ 2 \leq x_2 \leq 3 \end{array} \right\}$$

and define the following two functions on X:

$$u(x_1, x_2) = \begin{cases} -x_1 & 0 \leq x_2 \leq 1 \\ x_1 & 2 \leq x_2 \leq 3 \end{cases}$$

$$v(x_1, x_2) = \begin{cases} -x_1 & 0 \leq x_2 \leq 1 \\ x_1 + 1 & 2 \leq x_2 \leq 3 \end{cases}$$

Define $\gtrsim$ on X by maximization of v. As v is continuous, $\gtrsim$ satisfies axioms A1-A3. Note that u, restricted to $X^1 = \{ (x_1, x_2) \,|\, x_1 > 0 \}$, represents $\gtrsim$ as well. Indeed, it has a continuous extension to $X - u$ itself. However, it does not represent $\gtrsim$ on X^0, as u is constant on X^0 which isn't an equivalence class of $\gtrsim$ (say, $(2, 0) \succ (1, 0)$). $\square$